Combinatorial Counting - 3.2, 3.3 Permutations and Binomial Coefficients

Permutation is a bijection $\pi : X \to X$ on some finite X.

Example $X = \{1, 2, 3, 4, 5\}$, $\pi(1) = 3$, $\pi(2) = 5$, $\pi(3) = 4$, $\pi(4) = 1$, $\pi(5) = 2$.

or

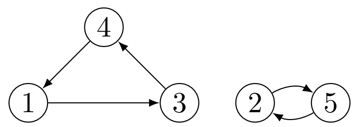
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

or

$$\begin{pmatrix} 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

or

As an *oriented graph* with vertices $[5]$ and oriented edges according to π.



Claim The oriented graph is a union of disjoint cycles. (notice that at each vertex, one arrow is going in and one is going out)

In cycle notation: $((1, 3, 4), (2, 5))$.

1: What is the number of permutations on $[n]$?

To minimize exceptions, notice we define $0! = 1$.

Binomial coefficient for $n, k \in \mathbb{N}_0$, $k \leq n$ is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

or

$$\binom{n}{k} = \frac{\prod_{i=0}^{k-1}(n-i)}{k!}$$

2: Show that $\binom{n}{k}$ counts the number of k-element subsets of n.
Plan A: Take a permutation of $[n]$ and first k entries are the subset. How many times is each subset counted?
Plan B: Take an injective mapping $[k] \to [n]$. How many times is each subset counted in the injective mappings?

Examples:

$$\binom{N}{0} = \qquad \binom{N}{N} = \qquad \binom{1}{1} =$$

$$\binom{8}{4} =$$

Notation: For a set X, we denote the set of all subsets of size k as

$$\binom{X}{k} = \{Y \subseteq X : |Y| = k\}$$

Notice that

$$\left|\binom{X}{k}\right| = \binom{|X|}{k}$$

3: Show the following identities

$$\binom{n}{k} = \binom{n}{n-k} \text{ and } \binom{n}{k} = \binom{n-1}{k-1} \cdot \frac{n}{k}$$

4: Show the following identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This identity is attributed to Pascal.

5: Show the Pascal's identity without expanding it using factorials. I.e. find a combinatorial argument.
Hint: Take an n-element set X and fix $a \in X$. Now count all subsets of size k but count sets that do contain a and that do not contain a separately.

Some identities can be explained using expansion of the factorials, but combinatorial arguments are often more elegant and provide some additional insight why certain identities hold.

Pascal's identity can be used to generate so called Pascal's triangle easily. It is a triangle, where in row n, binomial coefficients $\binom{n}{0} \cdots \binom{n}{n}$ are listed.

6: Fill Pascal's triangle, where row n lists all binomial coefficients $\binom{n}{k}$ for all k. Hint: Pascal's Formula.

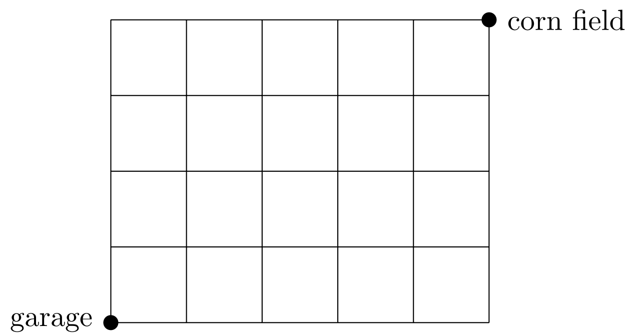
$n = 0$	$\binom{0}{0}$	1
$n = 1$	$\binom{1}{0}$ $\binom{1}{1}$	1 1
$n = 2$	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$	1 2 1
$n = 3$	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$	_ _ _ _
$n = 4$	$\binom{4}{0}$ _ _ _ _	_ _ _ _ _
$n = 5$	$\binom{5}{0}$ _ _ _ _ _	_ _ _ _ _ _
$n = 6$	_ _ _ _ _ _ _	_ _ _ _ _ _ _

7 *Committee approach***:** Prove that

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Think of finding a subset of a subset.

8 *Lattice paths***:** Count the number of ways drive a corn harvester from the garage (located at $0, 0$) to a corn field, located at $(5, 4)$.



Allowed movements are only right or up (you want to go the shortest path). (Try also counting in general to location (a, b), where $a, b \in \mathbb{Z}$)

9: Prove that

$$\sum_{0 \leq j \leq b} \binom{a+j-1}{a-1} = \binom{a+b}{a}$$

Hint: Use lattice paths.

10: Prove that

$$\sum_{0\le j\le n} \binom{n}{j}^2 = \binom{2n}{n}$$

Hint: $\binom{n}{j} = \binom{n}{m-j}$.

11: Prove that $\binom{a+b}{a}$ equals the number of partitions (a sequence of integers $P_1 \ge P_2 \ge \ldots \ge P_b$) satisfying $a \ge P_1 \ge P_2 \ge \ldots \ge P_b \ge 0$.
Hint: Try to enumerate all partitions for $a = b = 2$. It will help you understand the problem.

12 *Binary string*: Prove that

$$\sum_{0\le k\le n} \binom{n}{k} = 2^n$$

13: Prove that

$$\sum_{r\le k\le n} \binom{k}{r} = \binom{n+1}{r+1}$$

14: Prove

$$\sum_{0\le k\le r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$