Maximizing the number of odd cycles in a planar graph

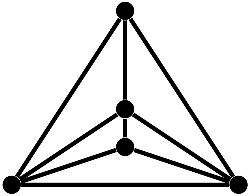
Emily Heath

Joint with Ryan Martin and Chris Wells

October 7, 2023

Planar graphs

A drawing of a graph in the plane without any crossing edges is called a **plane graph**. A graph with a plane graph drawing is a **planar graph**.



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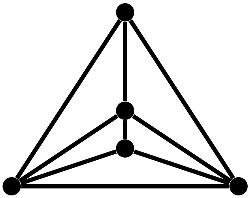

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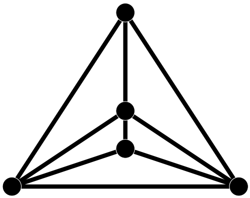

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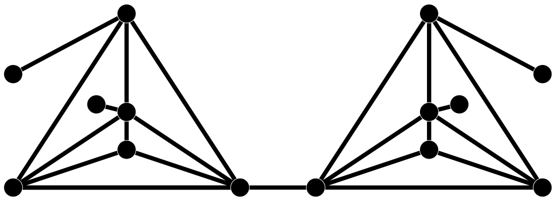

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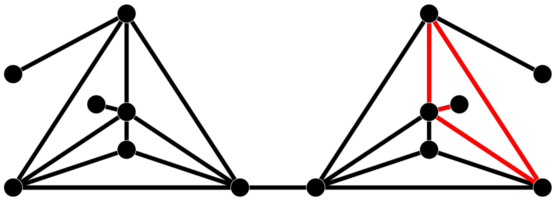

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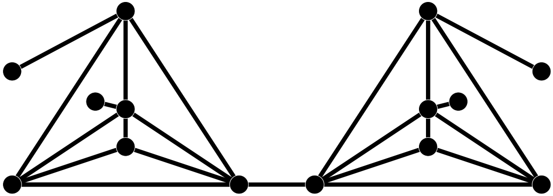

Lemma (Handshaking Lemma for plane graphs)

For every plane graph G, $2|E| = \sum_{v \in V} \deg(v) = \sum_{f \in F} \deg(f)$.

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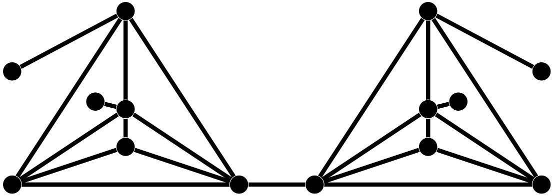

Lemma (Handshaking Lemma for plane graphs)

For every plane graph G, $2|E| = \sum_{v \in V} \deg(v) = \sum_{f \in F} \deg(f) \geq 3|F|$.

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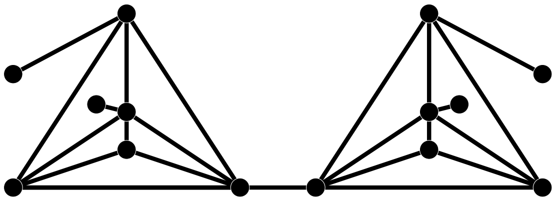

Lemma (Handshaking Lemma for plane graphs)

For every bipartite plane graph G, $2|E| = \sum_{v \in V} \deg(v) = \sum_{f \in F} \deg(f) \geq 4|F|$.

Euler's Theorem

Theorem (Euler 1758)

Let G be a plane graph with vertices V, edges E, and faces F. Then,

$$|V| - |E| + |F| = 2.$$

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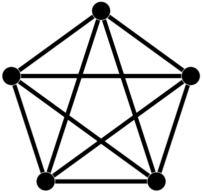
Corollary

For every plane graph G with $|V| \geq 3$,

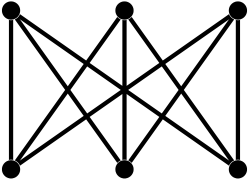
- $|E| \leq 3|V| - 6$
- $|E| \leq 2|V| - 4$, *if G is bipartite.*

Euler's Theorem

K_5



$|V| = 5,\ |E| = 10$



$K_{3,3}$

$|V| = 6,\ |E| = 9$

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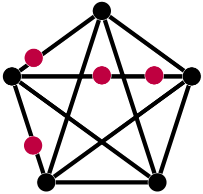

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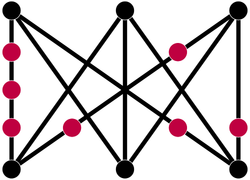

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Theorem

If G is a graph, then G is planar iff

- *G has no* **subdivision** *of K_5 or $K_{3,3}$; [Kuratowski, 1930]*
- *G has no minor of K_5 or $K_{3,3}$. [Wagner, 1937]*

An extremal problem

Let $\mathbf{N}_{\mathcal{P}}(n, H)$ denote the maximum number of copies of H in an n-vertex planar graph.

Theorem

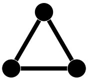
If $n \geq 3$, then $\mathbf{N}_{\mathcal{P}}(n, K_2) = 3n - 6$.

This is achieved by any planar triangulation.

Tutte showed that there are $\frac{n^{-7/2}}{64\sqrt{6\pi}}\left(\frac{256}{27}\right)^{n-2}$ planar triangulations.

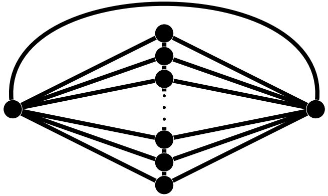
Hakimi-Schmeichel

Let $\mathbf{N}_{\mathcal{P}}(n, H)$ denote the maximum number of copies of H in an n-vertex planar graph.



Theorem (Hakimi-Schmeichel, 1979)

If $n \geq 3$, then $\mathbf{N}_{\mathcal{P}}(n, C_3) = 3n - 8$.



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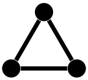

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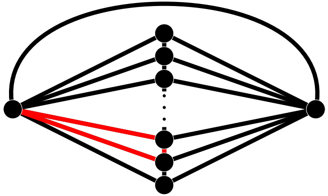

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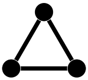

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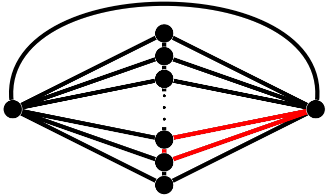

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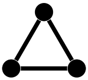

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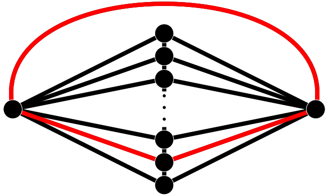

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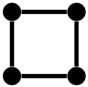
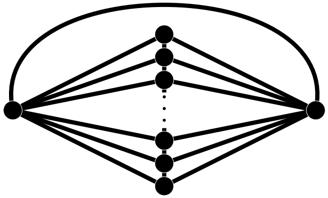


Theorem (Hakimi-Schmeichel, 1979)

If $n \geq 3$, *then* $\mathbf{N}_{\mathcal{P}}(n, C_4) = \dfrac{n^2 + 3n - 22}{2}$.



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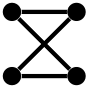

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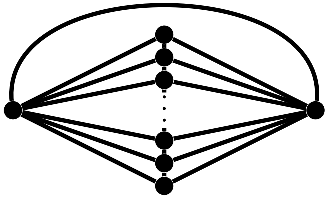
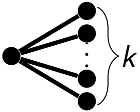
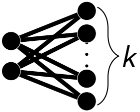


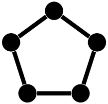
Other results

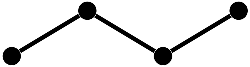
Alon and Caro (1984)



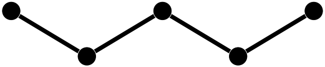


Győri, Paulos, Salia, Tompkins, Zamora (2019)

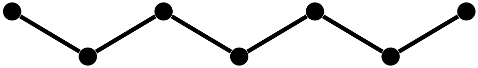




Ghosh, Győri, Martin, Paulos, Salia, Xiao, Zamora (2021)



Seven-paths and six-cycles



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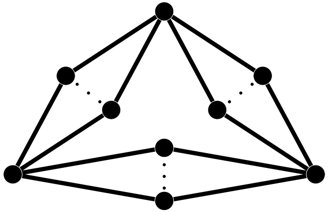
$$\mathbf{N}_{\mathcal{P}}(n, P_7) = \frac{4}{27}n^4 + O\left(n^{4-1/5}\right)$$



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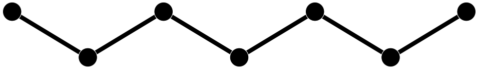

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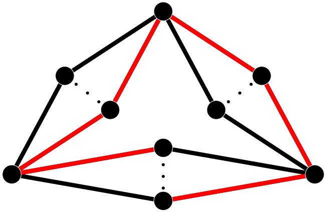

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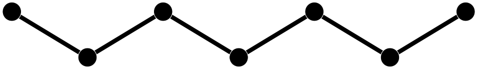

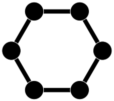

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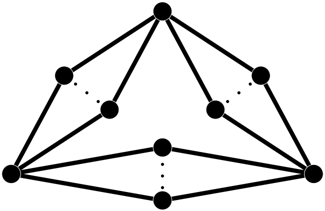

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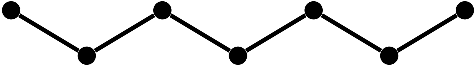

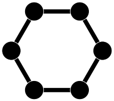

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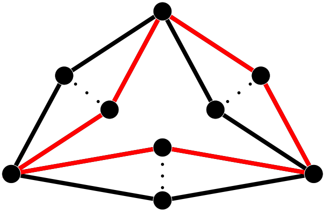

Even cycles

Theorem (Cox-Martin, 2022 & 2023)

$$\mathbf{N}_{\mathcal{P}}(n, C_8) = \left(\frac{n}{4}\right)^4 + O\left(n^{4-1/5}\right)$$
$$\mathbf{N}_{\mathcal{P}}(n, C_{10}) = \left(\frac{n}{5}\right)^5 + o\left(n^5\right)$$
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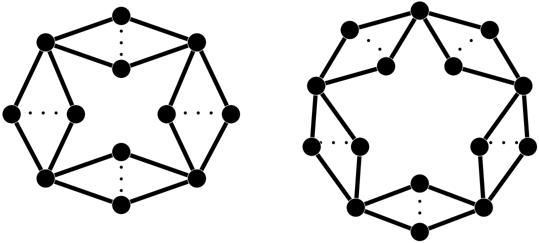

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If $m \geq 4$, then

$$4m\left(\frac{n}{m}\right)^{m+1} + O\left(n^m\right) \leq \mathbf{N}_{\mathcal{P}}(n, P_{2m+1}) \leq \frac{n^{m+1}}{2 \cdot (m-1)!} + O\left(n^{m+4/5}\right).$$

If $m \geq 5$, then

$$\left(\frac{n}{m}\right)^m + O\left(n^m\right) \leq \mathbf{N}_{\mathcal{P}}(n, C_{2m}) \leq \frac{n^m}{m!} + O\left(n^{m-1/5}\right).$$

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Theorem (Lv-Győri-He-Salia-Tompkins-Zhu, 2022)

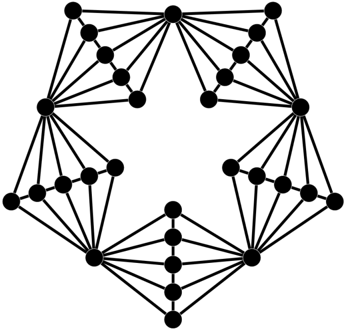
For all $k \geq 3$,

$$\mathbf{N}_{\mathcal{P}}(n, C_{2k}) = \left(\frac{n}{k}\right)^k + o\left(n^k\right).$$

Odd cycles

Theorem (H.-Martin-Wells, 2023)

$$\mathbf{N}_{\mathcal{P}}(n, C_5) = 2n^2 + O(n^{2-1/5})$$
$$\mathbf{N}_{\mathcal{P}}(n, C_{2m+1}) = 2m\left(\frac{n}{m}\right)^m + O(n^{m-1/5}) \text{ for } m \in \{3, 4\}$$
$$\mathbf{N}_{\mathcal{P}}(n, C_{2m+1}) \le 3m\left(\frac{n}{m}\right)^m + O(n^{m-1/5}) \text{ for } m \ge 5.$$



Proof idea

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Idea: Reduce the problem of bounding $\mathbf{N}_{\mathcal{P}}(n, H)$, assuming H has a special subdivision structure, to a maximum likelihood estimator question.

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Question

Which probability distribution μ on the edges of a clique maximizes the probability that $e(H')$ many edges sampled independently from μ yields a copy of H'?

Probability mass on a graph

Let μ be an edge probability measure for a clique K.

For each subgraph $H \subseteq K$, define $\mu(H) = \prod_{e \in E(H)} \mu(e)$.

For each graph H, define $\beta(\mu; H) = \sum_{H' \text{ a copy of } H \text{ in } K} \mu(H')$.

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Lemma (Reduction lemma for even cycles (Cox–Martin))

For every n-vertex planar graph G, there is an edge probability measure μ such that for all $m \geq 3$, the number of copies of C_{2m} in G is at most

$$\beta(\mu; C_m) \cdot n^m + O\left(n^{m-1/5}\right).$$

Probability mass on a graph

Let μ be an edge probability measure for a clique K.

For each subgraph $H \subseteq K$, define $\mu(H) = \prod_{e \in E(H)} \mu(e)$.

For each graph H, define $\beta(\mu; H) = \sum_{H' \text{ a copy of } H \text{ in } K} \mu(H')$.

Lemma (Reduction lemma for odd cycles (H.–Martin–Wells))

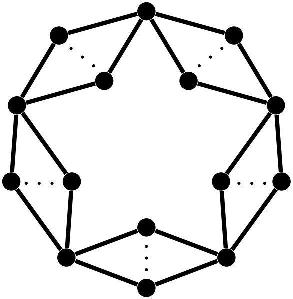
For every n-vertex planar graph G, there is an edge probability measure μ such that for all $m \geq 3$, the number of copies of C_{2m+1} in G is at most

$$(2m\beta(\mu; C_m) + \beta(\mu; P_{m+1})) \cdot n^m + O\left(n^{m-1/5}\right).$$

Reduction lemma for odd cycles

Proof idea: Given a planar graph G on n vertices, find a planar graph G' with the following properties:

- G' is highly structured
- G' and G have the same number of copies of C_{2m+1}, up to a small error term
- Counting the cycles C_{2m+1} in G' is asymptotically equivalent to solving our maximum likelihood problem



Reduction lemma for odd cycles

Proof idea: Given a planar graph G on n vertices, find a planar graph G' with the following properties:

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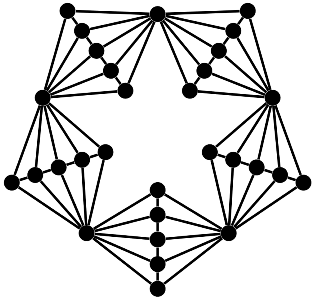

Open problems

- Improve the bound on the maximum likelihood question for C_{2m+1} when $m \geq 5$
- Prove an analogous reduction lemma for even paths
- For a given graph H, what is the μ (on some K_n for all values of n) that maximizes the probability of finding a copy of H by choosing $|E(H)|$ edges independently at random from μ?

Thank you!

Solving the maximum likelihood problem

Theorem (H.-Martin-Wells, 2023)

$$\sup_{\mu}\left(2\sum_{e\in\operatorname{supp}\mu}\mu(e)^2+\beta(\mu;P_3)\right)=2, \qquad \textit{and}$$

$$\sup_{\mu}\left(2m\cdot\beta(\mu;C_m)+\beta(\mu;P_{m+1})\right)=\frac{2}{m^{m-1}}, \qquad \textit{for } m\in\{3,4\}, \textit{ and}$$

$$\sup_{\mu}\left(2m\cdot\beta(\mu;C_m)+\beta(\mu;P_{m+1})\right)<\frac{2.7}{m^{m-1}}, \qquad \textit{for all } m\geq 5.$$

Proof ideas:

- KKT conditions
- AM-GM inequality
- Induction on number of vertices