Macaulay Posets and Rings

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What is this talk about?

1. Macaulay posets.
2. Macaulay rings.
3. An equivalence between the two.
4. Maybe applications of the equivalence.

The basic idea of the two objects

Macaulay Posets

Macaulay posets are posets in which an analog of the Kruskal-Katona Theorem holds.

Macaulay Rings

Macaulay rings are rings in which an analog of Macaulay's Theorem for lex ideals holds.

Objects in the Kruskal-Katona and Clements-Lindström Theorems

These theorems concern the hypercube/Boolean lattice/power set/grid.

Boolean Lattice

The set (for a fixed $d \in \mathbb{N}$)

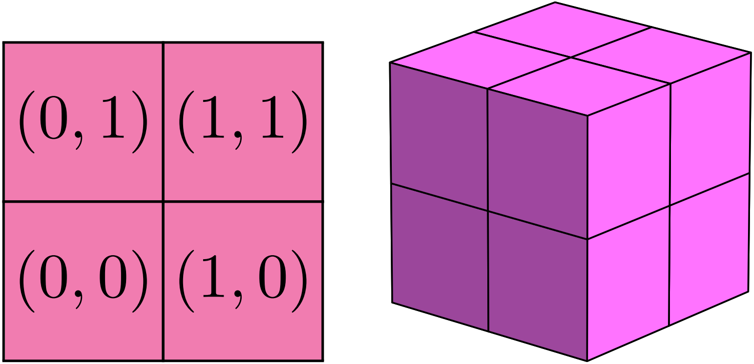
$$\{(a_1, \ldots, d_d) \in \mathbb{N}^d \mid a_i \in \{0, 1\}\} \cong \mathcal{P}(\{1, 2, \ldots, d\}).$$

Multiset lattices

Set of the form (for $d \in \mathbb{N}$ and $\ell_1, \ldots, \ell_d \in \mathbb{N} \cup \{\infty\}$)

$$\{(a_1, \ldots, d_d) \in \mathbb{N}^d \mid a_i < \ell_i\}.$$

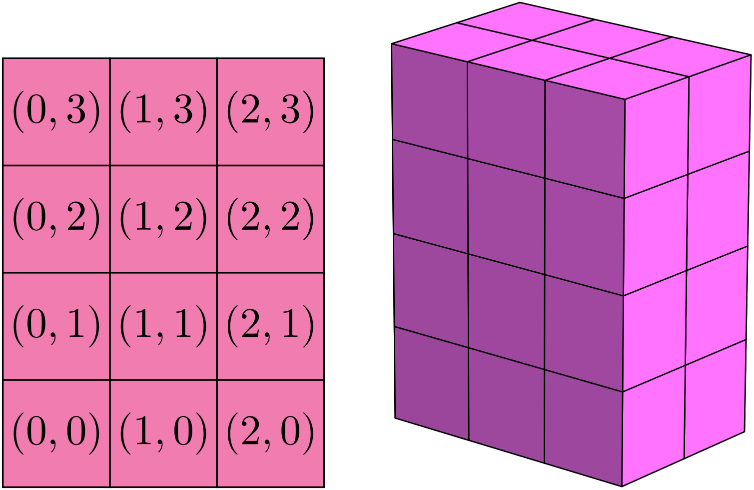
2D and 3D Boolean Lattices



(a) The 2D Boolean lattice.

(b) The 3D Boolean lattice.

2D and 3D Multiset Lattices



(a) The 2D multiset lattice with $\ell_1 = 3$ and $\ell_2 = 4$.

(b) The 3D multiset lattice with $\ell_1 = 2$, $\ell_3 = 4$ and $\ell_3 = 4$.

Main Components in the Kruskal-Katona and Clements-Lindström Theorems

Shadows

The **lower shadow**, $\Delta(x)$, of an element x is the set of elements right before that element. The **upper shadow**, $\nabla(x)$, of an element x is the set of elements right after that element.

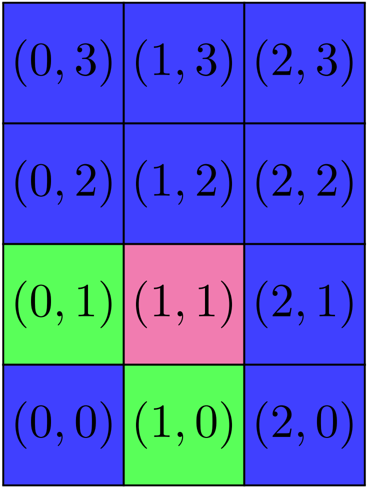
Initial Segment

Suppose that S is a finite set and $\mathcal{O}$ is a total order on S. For any $n \in \mathbb{N}$ the **initial segment** $\mathcal{O}[n]$ is the set of the first n elements of S under $\mathcal{O}$.

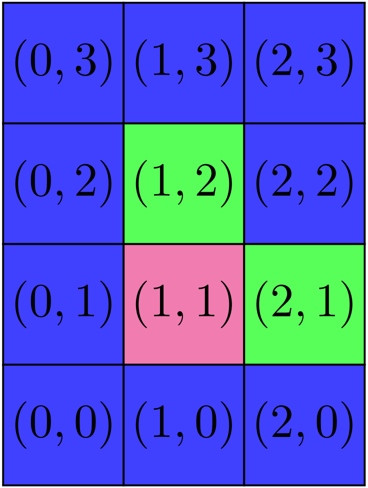
Lexicographic Order

For any $x, y \in \mathbb{N}^d$ we say that x is less than y in **lexicographic order** $(x <_{\mathcal{L}} y)$ iff for some $i \in \{1, \ldots, d-1\}$ we have $x_1 = y_1, \ldots, x_i = y_i$ and $x_{i+1} < y_{i+1}$.

Shadows

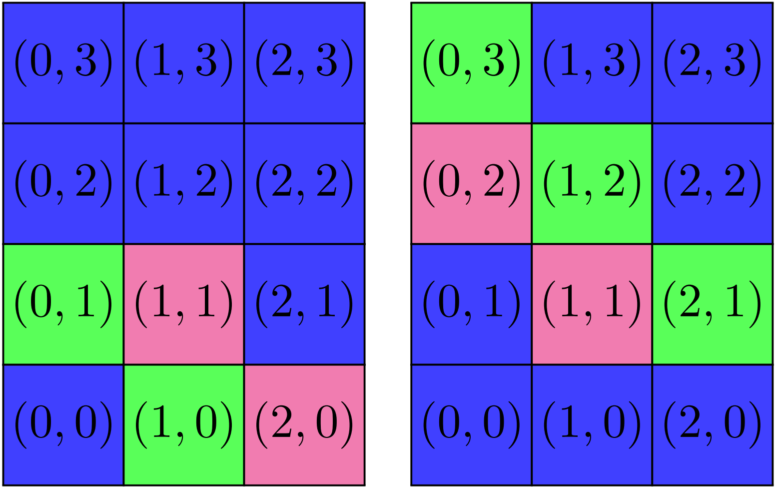


(a) $\Delta((1,1)) = \{(0,1),(1,0)\}$.



(b) $\nabla((1,1)) = \{(1,2),(2,1)\}$.

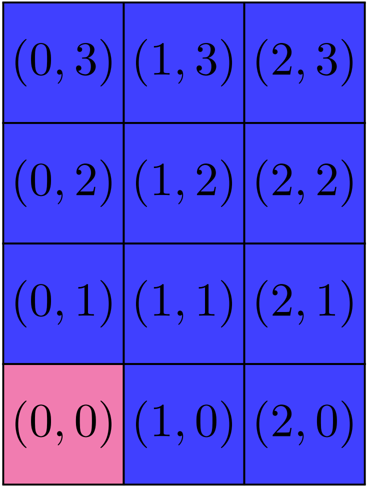
Shadows of Sets



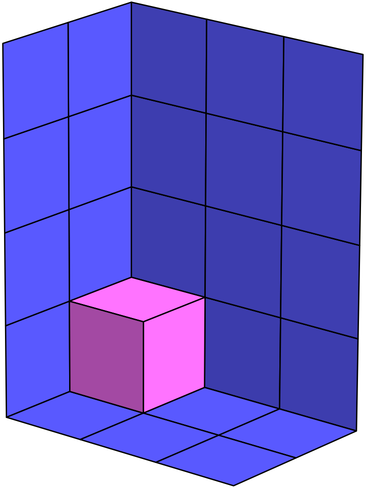
(a) $\Delta(\{(1,1),(2,0)\})$.

(b) $\nabla(\{(0,2),(1,1)\})$.

Lexicographic Order

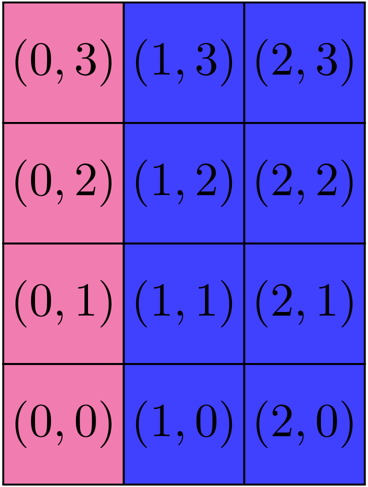


(a) Lexicographic Order in 2D.

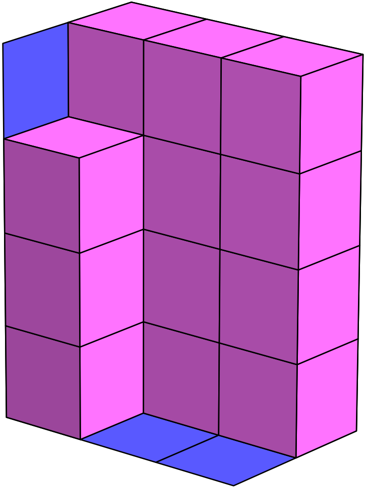


(b) Lexicographic Order in 3D.

Initial Segments



(a) $\mathcal{L}[4]$.



(b) $\mathcal{L}[15]$.

Levels and Ranks

Multiset lattices are ranked posets

Ranked Poset

A poset $\mathscr{P}$ is **ranked** if there exists a function $r : \mathscr{P} \to \mathbb{N}$, such that whenever we have $a \in \Delta(b)$ then $r(a) + 1 = r(b)$.

Levels

For $n \in \mathbb{N}$ we define the n-th **level**

$$\mathsf{Lvl}_n = \{x \in \mathscr{P} \mid r(x) = n\}.$$

All posets will be ranked and we will always talk about initial segments inside levels.

Initial Segments and Levels

$(0,5)$	$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$
$(0,4)$	$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$
$(0,3)$	$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$
$(0,2)$	$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$
$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$
$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$	$(5,0)$

(a) Lvl_5.

$(0,5)$	$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$
$(0,4)$	$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$
$(0,3)$	$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$
$(0,2)$	$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$
$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$
$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$	$(5,0)$

(b) $\mathcal{L}[3]$ in Lvl_5.

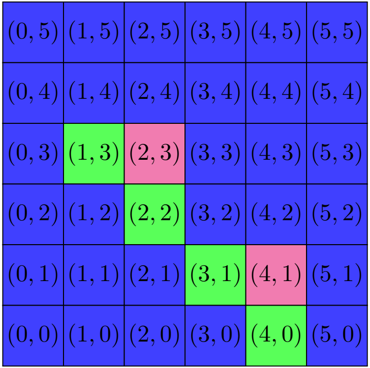
The Clements and Lindström Theorem

Theorem (Clements-Lindström 1969)

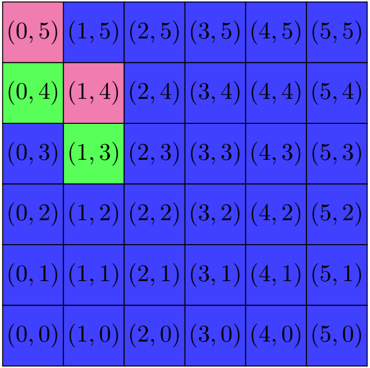
Suppose that $A \subseteq \mathsf{Lvl}_n$ is inside a multiset lattice with $\ell_1 \leq \cdots \leq \ell_d$. Then, where we write $\mathcal{L}[A]$ for the initial segment of $\mathcal{L}$ of size $|A|$,

1. $\Delta(\mathcal{L}[A])$ *is an initial segment of* $\mathcal{L}$.
2. $\Delta(A)$ *is at least as big as* $\Delta(\mathcal{L}[A])$.

Clements-Lindström in Action



(a) $A \subseteq \mathsf{Lvl}_5$ and $\Delta(A)$.



(b) $\mathcal{L}[A] \subseteq \mathsf{Lvl}_5$ and $\Delta(\mathcal{L}[A])$.

Macaulay Posets (Engel 1997)

For a ranked poset $\mathscr{P}$ be say that $\mathscr{P}$ is **Macaulay** if there exists a total order $\mathcal{O}$ on $\mathscr{P}$, such that for every $n \in \mathbb{N}$ and every $A \subseteq \mathsf{Lvl}_n$ we have

1. $\Delta(\mathcal{O}[A])$ is an initial segment of $\mathcal{O}$.
2. $\Delta(A)$ is at least as big as $\Delta(\mathcal{O}[A])$.

We say that $(\mathscr{P}, \mathcal{O})$ is Macaulay.

Theorem (Clements-Lindström)

Every multiset lattice is Macaulay, and we know a Macaulay ordering.

Main Components in Macaulay's Lex Ideal Theorems

Polynomial Ring

A field K and a polynomial ring $R = K[x_1, \ldots, x_d]$.

Monomials

A monomial is an element in R that has the form $x_1^{e_1} \cdots x_d^{e_d}$.

Graded Ideal

An ideal $I \subseteq R$ is graded if it can be written as

$$I = \bigoplus_{n=0}^{\infty} I_n,$$

where each I_n is a subspace of the vector space spanned by the monomials of degree n.

Polynomial Rings

Fields

Just think about K as $\mathbb{R}$.

Polynomial Rings

$R = K[x_1, \ldots, x_d]$ is the set of all polynomials with coefficients from K. Elements like $x_1 x_d$, $x_1 + x_d$, $x_1 x_d + x_1^{1000000000} x_d^3$.

Ideals

Ideals are subsets of $K[x_1, \ldots, x_d]$.

1. They have 0.
2. If f is in then $-f$ is in.
3. If you add two things you still end up in the ideal.
4. If f is in the ideal and $g \in K[x_1, \ldots, x_d]$ then fg is in the ideal.

Monomials Geometrically

$$
\begin{array}{|c|c|c|c|c}
\vdots & \vdots & \vdots & \vdots & \\
\hline
(0,3) & (1,3) & (2,3) & (3,3) & \dots \\
\hline
(0,2) & (1,2) & (2,2) & (3,2) & \dots \\
\hline
(0,1) & (1,1) & (2,1) & (3,1) & \dots \\
\hline
(0,0) & (1,0) & (2,0) & (3,0) & \dots \\
\hline
\end{array}
$$

(a) The 2D multiset lattice with $\ell_1 = \ell_2 = \infty$.

$$
\begin{array}{|c|c|c|c|c}
\vdots & \vdots & \vdots & \vdots & \\
\hline
x_1^0 x_2^3 & x_1^1 x_2^3 & x_1^2 x_2^3 & x_1^3 x_2^3 & \dots \\
\hline
x_1^0 x_2^2 & x_1^1 x_2^2 & x_1^2 x_2^2 & x_1^3 x_2^2 & \dots \\
\hline
x_1^0 x_2^1 & x_1^1 x_2^1 & x_1^2 x_2^1 & x_1^3 x_2^1 & \dots \\
\hline
x_1^0 x_2^0 & x_1^1 x_2^0 & x_1^2 x_2^0 & x_1^3 x_2^0 & \dots \\
\hline
\end{array}
$$

(b) Monomials in $R = K[x_1, x_2]$.

A Graded Ideal

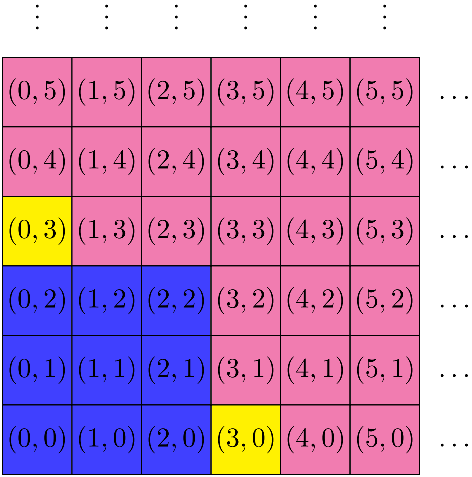


Figure: The ideal generated by x_1^3 and x_2^3 in $K[x_1, x_2]$.

The graded ideal algebraically

You need to have $I = \bigoplus_{n=0}^{\infty} I_n$, where I is generated by x_1^3 and x_2^3.

$$\begin{aligned}
I_0 &= \mathrm{Span}_K(0) \\
I_1 &= \mathrm{Span}_K(0) \\
I_2 &= \mathrm{Span}_K(0) \\
I_3 &= \mathrm{Span}_K(x_1^3, x_2^3) \\
I_4 &= \mathrm{Span}_K(x_1^4, x_1^3 x_2, x_1 x_2^3, x_2^4) \\
&\vdots
\end{aligned}$$

So, stuff like $x_1^3 + x_2^3$ and $x_1^3 + x_2^4$ is in the ideal, but NOT stuff like $\sum_{i=3}^{\infty} x_1^i$.

Hilbert Function

For a graded ideal $I = \bigoplus_{n=1}^{\infty} I_n \subseteq R = K[x_1, \dots, x_d]$ we define the Hilbert function of I to be $\mathrm{Hilb}_I : \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{Hilb}_I(n) = \dim_K I_n.$$

If $R = K[x_1, x_2]$ and $I = (x_1^3, x_2^3)$ then

$$\begin{aligned}
\mathrm{Hilb}_I(0) &= 0 \\
\mathrm{Hilb}_I(1) &= 0 \\
\mathrm{Hilb}_I(2) &= 0 \\
\mathrm{Hilb}_I(3) &= 2 \\
\mathrm{Hilb}_I(4) &= 4 \\
\mathrm{Hilb}_I(n) &= n+1 \text{ for all } n \geq 5
\end{aligned}$$

Dual Orders and Initial Lex Segment Spaces

Dual Order

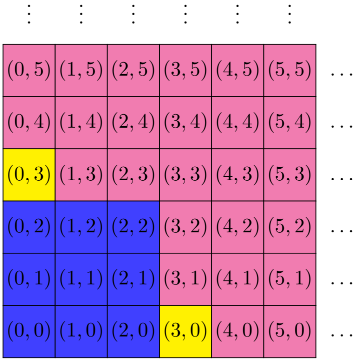
Suppose that S is a poset with a partial order $\mathcal{O}$. The dual order $\mathcal{O}^*$ is defined such that for $x, y \in S$ we have $x <_{\mathcal{O}^*} y$ iff $y <_{\mathcal{O}} x$.

Initial Lex Segment Space

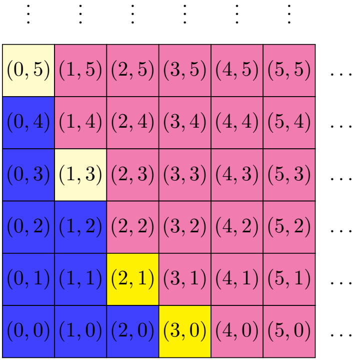
For a graded ideal $I = \bigoplus_{n=0}^{\infty} I_n \subseteq R$ we define the initial lex segment space of I to be

$$\mathcal{L}^*[I] = \bigoplus_{n=0}^{\infty} \text{Span}_K(\mathcal{L}^*[dim_K I_n])$$

I and $\mathcal{L}^*[I]$



(a) $I = (x_1^3, x_2^3)$.



(b) $\mathcal{L}^*[I]$.

Macaulay's Lex Ideal Theorem

Theorem (Macaulay 1927)

If I is a graded ideal in $R = K[x_1, \ldots, x_d]$ then

1. $\mathrm{Hilb}_I = \mathrm{Hilb}_{\mathcal{L}^*[I]}$.
2. $\mathcal{L}^*[I]$ *is a graded ideal.*

What do we need to generalize Macaulay's Theorem?

1. Different rings, not just $K[x_1, \ldots, x_d]$.
2. Monomials.
3. Graded Ideals and Hilbert function.
4. A total order on the set of monomials.

Different Rings and Monomials

For a graded ideal $H \subseteq R = K[x_1, \ldots, x_d]$ we are going to consider the quotient

$$S = R/H.$$

A monomial of S is a **nonzero** element of the form $x_1^{e_1} \cdots x_d^{e_d} + H$, and we say that it has degree $e_1 + \cdots + e_d$.

Graded Ideals and Hilbert Functions

Graded Ideals in Quotients

An ideal $I \subseteq S = R/H$ is graded if it can be written as

$$I = \bigoplus_{n=0}^{\infty} I_n,$$

where each I_n is a subspace of the vector space spanned by the monomials of degree n.

Hilbert Functions in Quotients

We define the Hilbert function of I to be $\mathrm{Hilb}_I : \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{Hilb}_I(n) = \dim_K I_n.$$

The Poset of Monomials $\mathscr{M}_S$ (K 2023+)

Monomial Partial Order

For two monomials m_1 and m_2 of S we say that $m_1 \leq m_2$ iff there exists a **monomial** m such that $m_1 m = m_2$.

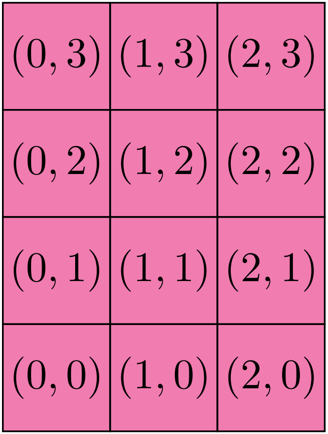
The Poset of Monomials

The above relation is a partial order and we can talk about the poset of monomials of the ring S, which will be denoted $\mathscr{M}_S$.

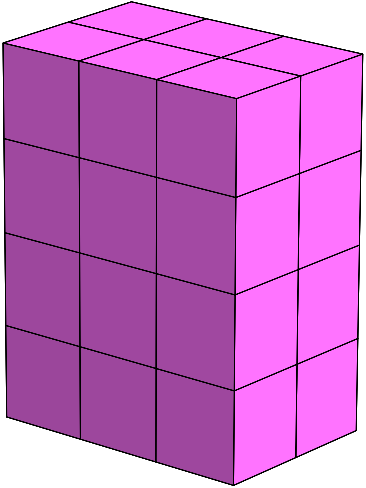
Ranks and Levels

The rank function of an element in $\mathscr{M}_S$ is given by its degree. Thus, we can talk about the levels of $\mathscr{M}_S$.

Some Posets of Monomials



(a) $\mathscr{M}_S$ with $\frac{K[x_1,x_2]}{(x_1^3,x_2^4)}$.



(b) $\mathscr{M}_S$ with $\frac{K[x_1,x_2,x_3]}{(x_1^2,x_2^3,x_3^4)}$.

Macaulay Rings (K 2023+)

Initial $\mathcal{O}$ Segment Space

Suppose that $\mathcal{O}$ is a total order on $\mathscr{M}_S$. For a graded ideal $I \subseteq S$ we define the initial $\mathcal{O}$ segment space of I to be

$$\mathcal{O}^*[I] = \bigoplus_{n=0}^{\infty} \mathrm{Span}_K(\mathcal{O}^*[\mathrm{Hilb}_I(n)])$$

Macaulay Ring

We say that S is Macaulay if there exists a total order $\mathcal{O}$ such that for every graded ideal $I \subseteq S$ we have:

1. $\mathcal{O}^*[I]$ is an ideal.
2. $\mathrm{Hilb}_I = \mathrm{Hilb}_{\mathcal{O}^*[I]}$.

We say that $(S, \mathcal{O})$ is Macaulay.

Macaulay's Theorem Restated

Theorem (Macaulay 1927)

$K[x_1, \ldots, x_d] \cong K[x_1, \ldots, x_d]/(0)$ *is Macaulay.*

A Macaulay Correspondence Theorem

Monomial Order

A total order $\mathcal{O}$ on $\mathscr{M}_S$ is a monomial order, if whenever we have $m_1, m_2, m \in \mathscr{M}_S$ with $m_1 < m_2$ then $mm_1 < mm_2$.

Theorem (K 2023+)

Suppose that:

1. *$\mathcal{O}$ is a total order on $\mathscr{M}_S$.*
2. *Every level of $\mathscr{M}_S$ is linearly independent.*
3. *There exists a monomial order on $\mathscr{M}_S$.*

Then $(S, \mathcal{O})$ is Macaulay if and only if $(\mathscr{M}_S, \mathcal{O})$ is Macaulay.

Main Ingredients in the Proof of the Macaulay Correspondence Theorem

Reducing from graded ideals to monomial ideals

Use the existence of a monomial order and level linear independence to obtain an ideal M that is generated by monomials and has the same Hilbert function.

Shadows and Multiplication

Upper shadows in $\mathscr{M}_S$ correspond to multiplying by the variables.

Bezrukov's Dual Lemma (Bezrukov 1994)

$(\mathscr{P}, \mathcal{O})$ is Macaulay if and only if $(\mathscr{P}^*, \mathcal{O}^*)$ is Macaulay.

Sum of Ideals $\Longleftrightarrow$ Tensor Product of Rings $\Longleftrightarrow$ Cartesian Product of Posets

Theorem (K 2023+)

Suppose that for all $i \in [d]$ we have $S_i = R_i/H_i$ for some graded ideal H_i of $R_i = K[x_{i,1}, \ldots, x_{i,n_i}]$, and that S_1 is level linearly independent. Let

$$S = \frac{K[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{d,1}, \ldots, x_{d,n_d}]}{(H_1 + H_2 + \cdots + H_d)}.$$

Then

1. $S_1 \otimes \cdots \otimes S_d \cong S$.
2. $\mathscr{M}_{S_1} \times \cdots \times \mathscr{M}_{S_d} \cong \mathscr{M}_S$.
3. $\mathscr{M}_S$ *is level linearly independent.*
4. *If there exist monomial orders for $\mathscr{M}_{S_1}, \ldots, \mathscr{M}_{S_d}$ then there is a monomial order on $\mathscr{M}_S$.*

Part 2

Get clever and apply this stuff.
But we can be done too :)

The Mermin-Murai Theorem on Colored Square Free Rings

Theorem (Mermin-Murai 2010)

Suppose that for all $i \in [d]$ we have $S_i = R_i/H_i$ with $H_i = (x_{i,1}, \ldots, x_{i,n_i})^2$ of $R_i = K[x_{i,1}, \ldots, x_{i,n_i}]$.

$$S = \frac{K[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{d,1}, \ldots, x_{d,n_d}]}{(H_1 + H_2 + \cdots + H_d)}.$$

Then S is Macaulay.

Decomposing the rings of Mermin and Murai

$$R_i = K[x_{i,1}, x_{i,2}, x_{i,3}]$$
$$H_i = (x_{i,1}, x_{i,2}, x_{i,3})^2 = (x_{i,1}^2, x_{i,2}^2, x_{i,3}^2, x_{i,1}x_{i,2}, x_{i,1}x_{i,3}, x_{i,2}x_{i,3})$$
$$S_i = R_i / H_i$$

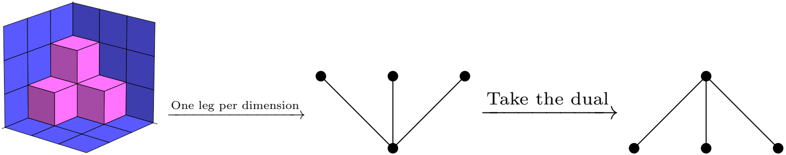


Figure: The poset of monomials for one of the rings in the tensor product

Star Posets

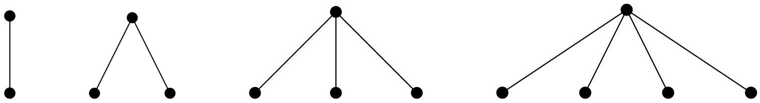


Figure: Hasse graphs of some basic star posets.

Star Posets

A star poset is a product of basic star posets.

Theorem (Many authors contributed 1971-1997)

All star posets are Macaulay.

The contributions to the star Macaulay Theorem

1. Lindström 1971.
2. Leeb 1978.
3. Bezrukov 1988.
4. Frankl, Füredi and Kalai 1988.
5. Bollobás and Radcliffe 1990.
6. Bollobás and Leader 1990.
7. London 1994
8. Leck 1995.
9. Engel 1997.

A new proof of the Mermin-Murai Theorem

1. Star posets are Macaulay.
2. By Bezrukov's Dual Lemma, duals of star posets are Macaulay.
3. Thus, the tensor product of the smaller rings in the Mermin-Murai Theorem is Macaulay.
4. Therefore, all colored-square free rings are Macaulay.

Spider Posets

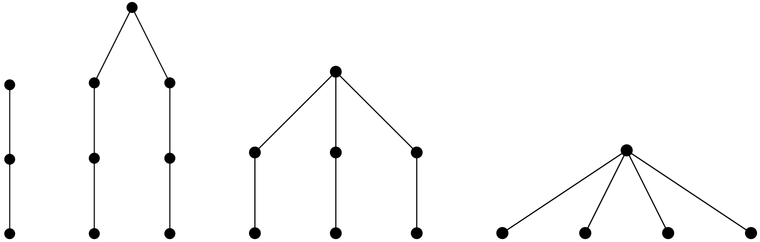


Figure: Hasse graphs of some spider posets.

Bezrukov-Elsässer poset

A *Bezrukov-Elsässer poset* is a poset of the form $\mathscr{P} \times \cdots \times \mathscr{P}$, where $\mathscr{P}$ is a spider.

Theorem (Bezrukov-Elsässer 2000)

All Bezrukov-Elsässer posets are Macaulay.

A Mermin-Murai Theorem for rings that are not square free

Bezrukov-Elsässer Rings

A *basic Bezrukov-Elsässer ring* has the form

$$S = \frac{K[x_1, \ldots, x_d]}{(x_1^\ell, \ldots, x_d^\ell) + (x_i x_j \mid i < j)}.$$

A *Bezrukov-Elsässer ring* has the form $S \otimes \cdots \otimes S$.

Theorem (K 2023+)

All Bezrukov-Elsässer rings are Macaulay.

Proof

Do the same thing you did for stars.

$$S = \frac{K[x_1, x_2, x_3]}{(x_1^3, x_2^3, x_3^3) + (x_i x_j \mid i < j)}.$$

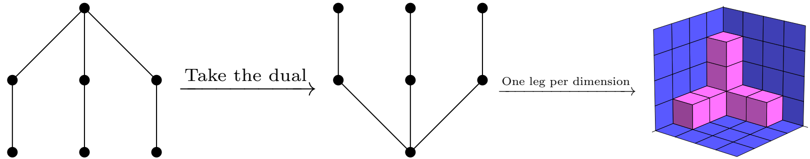


Figure: Relationship between spider posets and Bezrukov-Elsässer rings rings.

Tree Ring Classification

Tree Rings

We say that S is a *tree ring* if the Hasse graph of $\mathscr{M}_S$ is a tree.

Theorem (K 2023+)

Suppose that:

1. *S is a level linearly independent.*
2. *S is a tree ring.*
3. *$\mathscr{M}_S$ is finite.*
4. $n \geq \max_{a \in \mathscr{M}_S} r(a) + 3$

Then the n-fold product $S \otimes \cdots \otimes S$ is Macaulay if and only if $\mathscr{M}_S$ is isomorphic to the poset of monomials of a basic Bezrukov-Elsässer ring.

A Problem from Clements

Clements's Problem

Take a Macaulay poset $\mathscr{M}$ and another poset $\mathscr{P}$. When is $\mathscr{M} \times \mathscr{P}$ Macaulay? Clements could answer this question when all the elements in $\mathscr{P}$ are not comparable to each other.

The Bezrukov-Leck Problem, same as Clements's Problem

Can we do better than Clements? Why don't we try making $\mathscr{P}$ a chain? It is still hard. They were able to solve the problem when $\mathscr{M}$ has only two levels and the order is lexicographic.

The Mermin-Peeva and Shakin Theorem

Theorem

Suppose that $R = K[x_1, \dots, x_d]$ and let M be a ideal generated by monomials. If R/M is Macaulay with the lexicographic order, then $R[x_{d+1}]/(M)$ is Macaulay with the lexicographic order.

Corollary (K 2023+)

Let $\mathscr{M}_S$ be the poset of monomials for the ring R/M above. If $\mathscr{M}_S$ is Macaulay with the lexicographic order, then $\mathscr{M}_S \times \mathbb{N}$ is Macaulay with the lexicographic order.

Proof.

$$\frac{R[x_{d+1}]}{(M)} \cong \frac{R}{M} \otimes \frac{K[x_{d+1}]}{(0)}.$$

$\square$

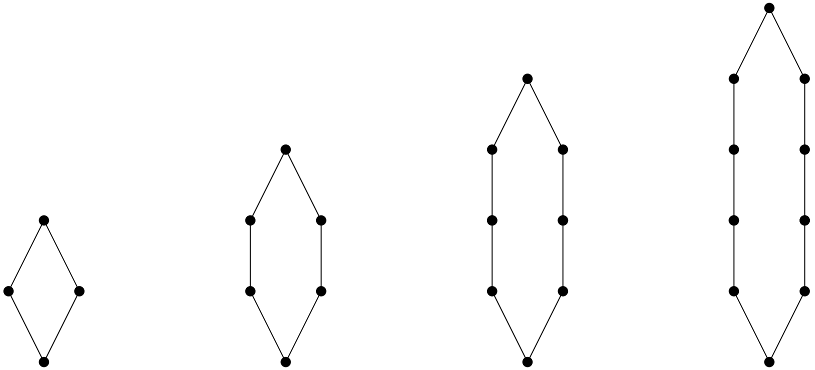
Let's do one more class for fun.

Consider rings of the form $S_1 \otimes \cdots \otimes S_d$, where

$$S_i = \frac{K[x_{i,1}, x_{i,2}]}{(x_{i,1}^{\ell_i}, x_{i,2}^{\ell_i}, x_{i,1}x_{i,2}, x_{i,1}^{\ell_i - 1} - x_{i,2}^{\ell_i - 1})}.$$

What does the poset of monomials of S_i look like?

Discrete Even Tori



The Karakhanyan-Riordan Theorem

1. Karakhanyan in 1982 solved the vertex isoperimetric problem for products of discrete tori.
2. Riordan in 1998 solved the same problem as Karakhanyan.
3. Bollobás and Leader 1990 solved the problem for some products.

Corollary

All products of discrete even tori are Macaulay.

Corollary (K 2023+)

$S_1 \otimes \cdots \otimes S_d$ *is Macaulay, where*

$$S_i = \frac{K[x_{i,1}, x_{i,2}]}{(x_{i,1}^{\ell_i}, x_{i,2}^{\ell_i}, x_{i,1}x_{i,2}, x_{i,1}^{\ell_i - 1} - x_{i,2}^{\ell_i - 1})}.$$

This is the first class of Macaulay rings that are not quotients by a monomial or toric ideal.

There is more stuff, but I got tired at this point

Thank you!