Longest Path Transversals in Chordal Graphs

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joint work with James Long and Kevin Milans

AMS Fall Central Sectional Meeting 2023

October 7, 2023

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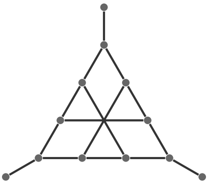
- One such is the Petersen fragment:



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- Bipartite Permutation Graphs (Cerioli et al. 2020)

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- Does there exist constant C, such that for all connected G, $lpt(G) \leq C$? (Walther 1969)

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- Balister, Györi, Lehel, and Schelp then asked if the larger family of chordal graphs have the Gallai property.

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- We achieved this bound by exploiting the tree-like structure of chordal graphs.

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- For a connected graph G, $lpt(G) \leq tw(G) + 1$ (Rautenbach and Sereni 2014)

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- Subtrees having the Helly property often translates to nice properties for chordal graphs.

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Lemma (Jordan 1869)

Let T be a tree, then there exists a vertex $z \in V(T)$ such that each component of $T - z$ has at most $|V(T)|/2$ vertices.

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- We recurse on subtrees of size at most $|V(T)|/2$.
- As chordal graphs have at most n maximal cliques, i.e. $|V(T)| \leq n$, this obtains a transversal of size $O(\log^2 n)$.

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- This saves a log factor from the size guarantee of the transversal.

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