ORDERED RAMSEY NUMBERS OF LOOSE PATHS AND MATCHINGS

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Abstract. For a k-uniform hypergraph G with vertex set $\{1,\dots,n\}$, the ordered Ramsey number $\mathrm{OR}_t(G)$ is the least integer N such that every t-coloring of the edges of the complete k-uniform graph on vertex set $\{1,\dots,N\}$ contains a monochromatic copy of G whose vertices follow the prescribed order. Due to this added order restriction, the ordered Ramsey numbers can be much larger than the usual graph Ramsey numbers. We determine that the ordered Ramsey numbers of loose paths under a monotone order grows as a tower of height two less than the maximum degree in terms of the number of edges. We also extend theorems of Conlon, Fox, Lee, and Sudakov [Ordered Ramsey numbers, arXiv:1410.5292] on the ordered Ramsey numbers of 2-uniform matchings to provide upper bounds on the ordered Ramsey number of k-uniform matchings under certain orderings.

1. Introduction

Ramsey theory is a fundamental topic in extremal graph theory. The Ramsey number $\mathrm{R}_t(n)$ is the minimum N such that every t-coloring of the edges of the complete graph of order N contains a monochromatic clique of order n. The number $\mathrm{R}_t(n)$ can also be defined as the maximum N such that there exists a t-coloring of K_{N-1} that avoids monochromatic copies of the graph K_n. This concept naturally generalizes to avoiding monochromatic copies of any k-uniform hypergraph G, defining the graph Ramsey number $\mathrm{R}_t(G)$, leading to a large number of available questions. The asymptotic growth of $\mathrm{R}_t(G)$ varies significantly, and depends on several properties of G, such as maximum degree [2] or degeneracy [10].

A recent variation, called *ordered Ramsey theory*, has received significant attention [1, 4, 6, 9, 12, 13]. In this variation, we again look for t-colorings of the complete graph that avoid monochromatic copies of a graph G, except that the *order* of the vertices of G in this monochromatic copy are very important. This modification relaxes some of the constraints on the coloring, so the ordered Ramsey numbers can be much larger than the usual graph Ramsey number, but is still bounded from above by the Ramsey number $\mathrm{R}_t(n)$ where n is the number of vertices in G. If G is a 2-uniform path under the standard ordering, then the 2-color ordered Ramsey number of G is equal to the bound of the Erdős-Szekeres Theorem [8] (see [3, 12]). If G is a tight 3-uniform path under the standard ordering, then the 2-color ordered Ramsey number of G is equal to the bound of the *happy ending problem* (see [9]). Due to these connections, much of the previous work has focused on the ordered Ramsey number of tight k-uniform paths under the standard ordering [9, 12, 13], but others considered 2-uniform matchings with an arbitrary ordering [1, 6]. We extend these investigations by determining strong bounds on the ordered Ramsey number of loose k-uniform paths and k-uniform matchings, using an arbitrary number of colors.

An *ordered k-uniform hypergraph* is a hypergraph G where the edge set $E(G)$ contains k-sets of vertices, and the vertex set $V(G)$ is totally ordered. An ordered hypergraph G is *contained* in an ordered hypergraph H if there is an injective, order-preserving map from the vertices of G to the vertices of H such that edges of G map to edges of H. Let K_N^k be the complete k-uniform hypergraph on the vertex set $\{1,\dots,N\}$ and let $c: E(K_N^k) \to \{1,\dots,t\}$ be a t-coloring of the edges

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in K_N^k. The i-colored subgraph of K_N^k is the ordered hypergraph given by the edges in $c^{-1}(i)$. For ordered k-uniform hypergraphs $G_1, \ldots, G_t$, the *ordered Ramsey number* $\mathrm{OR}(G_1, \ldots, G_t)$ is the minimum N such that for every t-coloring of K_N^k there is some color i such that the i-colored subgraph contains G_i. This number is necessarily defined and finite, since there exists an n such that each G_i is a subgraph of K_n^k and hence $\mathrm{OR}(G_1, \ldots, G_t) \leq \mathrm{R}_t(n)$. If $G_1 = \cdots = G_t = G$, then we denote $\mathrm{OR}(G_1, \ldots, G_t)$ as $\mathrm{OR}_t(G)$ and refer to this as the *diagonal case*; otherwise it is the *off-diagonal case*.

For positive integers k, ℓ, e such that $k > \ell$, the *(k,ℓ)-path on e edges*, denoted $P_e^{k,\ell}$, is the k-uniform ordered hypergraph on $e(k-\ell)+\ell$ vertices and e totally-ordered edges $A_1, A_2, \ldots, A_e$ where two consecutive edges A_i, A_{i+1} intersect exactly on the maximum ℓ vertices in A_i and the minimum ℓ vertices in A_{i+1}. The path $P_e^{k,k-1}$ is called the *tight k-uniform path* and otherwise $P_e^{k,\ell}$ is a *loose path*. For $\ell = 0$, we can extend the definition of $P_e^{k,\ell}$ by requiring that two consecutive edges A_i, A_{i+1} satisfy $\max A_i < \min A_{i+1}$, and hence the edges are disjoint, forming a *matching*. Note that when $k = 2$ the only possibilities are a tight path or a matching. We will primarily use the ordering given by this definition, and we will specify the special cases when we will consider a possibly different ordering on $P_e^{k,\ell}$.

Define the *intersection number*, $i(k,\ell)$, to be the maximum degree of a vertex in $P_e^{k,\ell}$ for all $e \geq k$. Observe that if $\ell > 0$, then $i(k,\ell)$ is the unique integer $m \geq 2$ that satisfies

$$\frac{m-2}{m-1} < \frac{\ell}{k} \leq \frac{m-1}{m}.$$

The tight paths $P_e^{k,k-1}$ have been investigated thoroughly. For 2-uniform tight paths, the ordered Ramsey number $\mathrm{OR}_t(P_e^{2,1})$ is determined by Choudum and Ponnusamy [3]. Further, Fox, Pach, Sudakov, and Suk [9] determined the growth of $\mathrm{OR}_t(P_e^{3,2})$ to be exponential in e and doubly-exponential in t, and Moshkovitz and Shapira [13] found that $\mathrm{OR}_t(P_e^{k,k-1})$ grows as a tower of height $k-2$ in e and as a tower of height $k-1$ in t. In fact, Moshkovitz and Shapira determine $\mathrm{OR}_t(P_e^{k,k-1})$ exactly in terms of high-dimensional integer partitions. In Section 2, we demonstrate an exact relationship between the ordered Ramsey numbers of loose paths and of tight paths of appropriate size.

Theorem 1.1. *For $k > \ell \geq 1$, $i = i(k,\ell)$, and positive integers $e_1, \ldots, e_t$,*

$$\mathrm{OR}(P_{e_1}^{k,\ell}, \ldots, P_{e_t}^{k,\ell}) = (k-\ell)\,\mathrm{OR}(P_{e_1}^{i,i-1}, \ldots, P_{e_t}^{i,i-1}) + \ell - (k-\ell)(i-1).$$

Therefore, the asymptotic growth of $\mathrm{OR}_t(P_e^{k,\ell})$ is a tower of height $i(k,\ell)-2$ in e and a tower of height $i(k,\ell)-1$ in t. In particular, when $i(k,\ell) = 2$ we can use the exact theorem for 2-uniform tight paths to exactly determine the ordered Ramsey number.

Corollary 1.2. *For $0 < 2\ell \leq k$ and positive integers $e_1, \ldots, e_t$,*

$$\mathrm{OR}(P_{e_1}^{k,\ell}, \ldots, P_{e_t}^{k,\ell}) = (k-\ell)\prod_{i=1}^{t} e_i + \ell.$$

Conlon, Fox, Lee, and Sudakov [6] and Balko, Cibulka, Král, and Kynčl [1] independently investigated how the ordered Ramsey number $\mathrm{OR}_t(G)$ differs among orderings of a 2-uniform graph G. In particular, they investigated upper bounds of $\mathrm{OR}_t(M)$ for a 2-uniform matching M, and found that these upper bounds are nearly sharp. In Section 3, we extend the methods in these papers to attain upper bounds on the ordered Ramsey numbers of k-uniform matchings under certain "controlled" orderings. We present an upper bound on the t-color ordered Ramsey number $\mathrm{OR}_t(P_e^{2,1})$ for an arbitrarily-ordered copy of $P_e^{2,1}$ that nearly matches the upper bound on $\mathrm{OR}_t(M)$ for a 2-uniform matching M, extending work of Cibulka, Gao, Krčál, Valla, and Valtr [4] on two colors. Several conjectures and open problems are presented in Section 4.

1.1. **Notation.** We follow standard notation from [15]. For an (ordered) hypergraph G, we use $V(G)$ as the vertex set of G, $E(G)$ as the edge set of G, $|G|$ as the number of edges in G, and k will always denote the size of an edge in G. For integers $m \leq n$, let $[n] = \{1, \ldots, n\}$, $[m, n] = \{m, m+1, \ldots, n-1, n\}$, and let $\binom{[n]}{m}$ denote the set of m-element subsets of $[n]$. For $k \geq 2$, the complete k-uniform (ordered) hypergraph with vertex set $[N]$ is denoted K_N^k. The 2-uniform case is special, so K_N denotes K_N^2.

We use $\lg n = \log_2 n$. We always use e the number of edges in a graph and never as the base of the natural logarithm. The *tower function of height* t, denoted by $\mathrm{tow}_t(n)$, is

$$\mathrm{tow}_0(n) = n, \quad \text{and} \quad \mathrm{tow}_t(n) = 2^{\mathrm{tow}_{t-1}(n)} \text{ for } t \geq 1.$$

2. Ordered Ramsey Numbers of Loose Paths

Proof of Theorem 1.1. Let $i = i(k, \ell)$ and $\ell' = \ell - (k-\ell)(i-2)$. Let $N = \mathrm{OR}(P_{e_1}^{i,i-1}, \ldots, P_{e_t}^{i,i-1})$ and $N' = (k-\ell)N + \ell'$.

For a k-uniform edge $\{x_1, \ldots, x_k\}$, we define the *rational reduction*, denoted $\underline{r}(x_1, \ldots, x_k)$, to be the the i-uniform edge $\{\lceil x_1/(k-\ell)\rceil, \lceil x_{(k-\ell)+1}/(k-\ell)\rceil, \ldots, \lceil x_{(i-1)(k-\ell)+1}/(k-\ell)\rceil\}$. For an i-uniform edge $\{x_1, \ldots, x_i\}$, the *canonical preimage*, denoted $\underline{r}^{-1}(x_1, \ldots, x_i)$, is defined as

$$\underline{r}^{-1}(x_1, \ldots, x_i) = \left[\bigcup_{j=1}^{i-1} \bigcup_{a=1}^{k-\ell} \{(k-\ell)(x_j - 1) + a\}\right] \cup \left[\bigcup_{a=1}^{\ell'} \{(k-\ell)(x_i - 1) + a\}\right].$$

Observe that $(i-1)(k-\ell) + \ell' = k$ and hence $\underline{r}^{-1}(x_1, \ldots, x_i)$ has k ordered elements. Finally, note that $\underline{r}$ sends k-uniform edges from $K_{N'}^k$ to i-uniform edges in K_N^i and $\underline{r}^{-1}$ sends i-uniform edges from K_N^i to k-uniform edges in $K_{N'}^k$.

Lower Bound. There exists a t-coloring $c : E(K_{N-1}^i) \to [t]$ of K_{N-1}^i that avoids a j-colored copy of $P_{e_j}^{i,i-1}$ for each $j \in [t]$. Define a coloring $c' : E(K_{N'-1}^k) \to [t]$ by $c'(x_1, \ldots, x_k) = c(\underline{r}(x_1, \ldots, x_k))$. Suppose that there is a color j and a list $x_1 < \cdots < x_m$ of vertices such that there is a j-colored copy of $P_{e_j}^{k,\ell}$ in c' on the vertices $x_1, \ldots, x_m$. Then, for each k-uniform edge $\{x_p, \ldots, x_{p+k-1}\}$ in this copy of $P_{e_j}^{k,\ell}$, the edge $\underline{r}(x_p, \ldots, x_{p+k-1})$ has color j in c. Also, for two consecutive edges $\{x_p, \ldots, x_{p+k-1}\}$ and $\{x_{p+k-\ell}, \ldots, x_{p+2k-\ell-1}\}$ the rational reductions $\underline{r}(x_p, \ldots, x_{p+k-1})$ and $\underline{r}(x_{p+\ell}, \ldots, x_{p+k+\ell-1})$ intersect in $i-1$ vertices. Thus, the e_j edges given by the rational reductions form a j-colored copy of $P_{e_j}^{i,i-1}$, a contradiction. Therefore, c' avoids a j-colored copy of $P_{e_j}^{k,\ell}$ and hence $\mathrm{OR}(P_{e_1}^{k,\ell}, \ldots, P_{e_t}^{k,\ell}) \geq N'$.

Upper Bound[1]. Let $c' : E(K_{N'}^k) \to [t]$ be a t-coloring of $K_{N'}^k$. Define a t-coloring $c : E(K_N^i) \to [t]$ of K_N^i as $c(\{x_1, x_2, \ldots, x_i\}) = c'(\underline{r}^{-1}(x_1, \ldots, x_i))$. By the definition of N, there exists a j-colored copy of $P_{e_j}^{i,i-1}$ on vertices $x_1, \ldots, x_m$ for some $j \in [t]$. For each i-uniform edge $\{x_q, \ldots, x_{q+i-1}\}$ in this copy of $P_{e_j}^{i,i-1}$, the k-uniform edge $\underline{r}^{-1}(x_q, \ldots, x_{q+i-1})$ also has the color j with respect to c'. Further, for two consecutive i-uniform edges $\{x_q, \ldots, x_{q+i-1}\}$ and $\{x_{q+1}, \ldots, x_{q+i}\}$ in this copy of $P_{e_j}^{i,i-1}$, the k-uniform edges $\underline{r}^{-1}(x_q, \ldots, x_{q+i-1})$ and $\underline{r}^{-1}(x_{q+1}, \ldots, x_{q+i})$ intersect in exactly ℓ vertices. Therefore, there is a j-colored copy of $P_{e_j}^{k,\ell}$ with respect to the coloring c' and therefore $\mathrm{OR}(P_{e_1}^{k,\ell}, \ldots, P_{e_t}^{k,\ell}) \leq N'$. □

The bounds of Moshkovitz and Shapira on $\mathrm{OR}_t(P_e^{k,k-1})$ [13, Corollary 3] imply the following.

[1] The authors thank Josef Cibulka for providing the translation of colorings in this direction.

Corollary 2.1. *For $e \geq 2$, $k < 2\ell < 2k$, and $\ell' = \ell - (k-\ell)(i(k,\ell)-1)$,*

$$(k-\ell)\operatorname{tow}_{i(k,\ell)-2}(e^{t-1}/2\sqrt{t}) + \ell' \leq \mathrm{OR}_t(P_e^{k,\ell}) \leq (k-\ell)\operatorname{tow}_{i(k,\ell)-2}(2e^{t-1}) + \ell'.$$

In [11], Gerencsér and Gyárfás showed that for $n \geq m \geq 1$,

$$\mathrm{R}(P_n^{2,1}, P_m^{2,1}) = n + \left\lfloor \frac{m}{2} \right\rfloor + 2.$$

Comparatively, $\mathrm{OR}(P_n^{2,1}, P_m^{2,1}) = nm+1$, which shows a large discrepancy between the ordered and unordered variants of the Ramsey number in just the 2-uniform case. It should, however, be noted that over all orderings of a (k,ℓ)-path, the standard ordering on $P_e^{k,\ell}$ does not necessarily minimize the ordered Ramsey number. For example, it is easy to observe that there exists an ordering of $P_2^{k,k-1}$ such that $\mathrm{OR}_t(P_2^{k,k-1}) \leq k+t$.

Now that we have determined the ordered Ramsey number for a particularly "nice" ordering of a (k,ℓ)-path, it is natural to ask for general bounds on $\mathrm{OR}_t(P_e^{k,\ell})$ where the vertices of $P_e^{k,\ell}$ are ordered arbitrarily. In this direction Cibulka, Gao, Krčál, Valla, and Valtr [4, Theorem 6] prove that if the vertices of $P_e^{2,1}$ are arbitrarily ordered then $\mathrm{OR}_2(P_e^{2,1}) \leq 2^{O(\lg^2 e)}$. Their proof technique can easily be extended to show that $\mathrm{OR}_t(P_e^{2,1}) \leq 2^{O(\lg^t e)}$.

As a means to a lower bound on this value, Conlon, Fox, Lee and Sudakov [6] provided the following lower bound on the ordered Ramsey number of a randomly-ordered 2-uniform matching, which was also proved in a weaker form by Balko, Cibulka, Král and Kynčl [1].

Theorem 2.2 (Conlon, Fox, Lee and Sudakov [6, Theorems 2.3])**.** *There exists a positive constant c, such that if M is a randomly-ordered matching on e edges, then asymptotically almost surely,*

$$\mathrm{OR}_2(M) \geq (2e)^{c\log(2e)/\log\log(2e)}.$$

Since $P_e^{2,1}$ contains a matching of size $\lceil e/2 \rceil$, we see that almost every ordering of $P_e^{2,1}$ yields $\mathrm{OR}_2(P_e^{2,1}) \geq 2^{\Omega(\lg^2 e/\lg\lg e)}$. Hence, the result of Cibulka, Gao, Krčál, Valla, and Valtr is fairly tight when $t=2$. Therefore, for almost every ordering of $P_e^{2,1}$, $\mathrm{OR}_t(P_e^{2,1})$ grows as a quasi-polynomial in e for a fixed t and possibly double-exponentially in t for a fixed e. Comparatively, for the standard ordering of $P_e^{2,1}$, $\mathrm{OR}_t(P_e^{2,1})$ grows polynomially in e and exponentially in t.

The techniques used by Cibulka, Gao, Krčál, Valla, and Valtr do not seem to extend to higher uniformities in any meaningful way, so finding an upper bound on $\mathrm{OR}_t(P_e^{k,\ell})$ for an arbitrary ordering of $P_e^{k,\ell}$ will require new techniques.

3. Ordered Ramsey Numbers of k-Uniform Matchings

Recall that the ordered path $P_e^{k,0}$ has disjoint edges, and therefore is a matching. We will consider a more general class of ordered matchings.

For a fixed $0 \leq r \leq k$ and positive integer e, the (k,r)-*nested matching on e edges* is the ordered graph $M_e^{k,r}$ defined iteratively as: $E(M_1^{k,r})$ consists of one edge $A_1 = [k]$, and $E(M_{e+1}^{k,r})$ consists of the edges in $E(M_e^{k,r})$ and an edge A_{e+1} consisting of the r least integers greater than $\max V(M_e^{k,r})$ and the $k-r$ greatest integers less than $\min V(M_e^{k,r})$. We say (k,r) is the *nesting pattern* of $M_e^{k,r}$. Note that $M_e^{k,r}$ is isomorphic to $M_e^{k,k-r}$ when the ordering is reversed, and $M_e^{k,0} \cong M_e^{k,k} \cong P_e^{k,0}$.

In [5], Cockayne and Lorimer show that for integers $e_1 \geq \cdots \geq e_t$, if M_i is a 2-uniform matching on e_i edges, then

$$\mathrm{R}(M_1, \ldots, M_t) = e_1 + 1 + \sum_{i=1}^{t}(e_i - 1).$$

This value is not far from the value of the ordered Ramsey number for 2-uniform nested matchings. The following lemma presents a lower bound on the ordered Ramsey number of t k-uniform nested matchings, even if the nesting patterns differ among the matchings.

Lemma 3.1. *For positive integers $e_1, \dots, e_t$ and $r_1, \dots, r_t \in \{0, \dots, k\}$,*

$$\mathrm{OR}(M_{e_1}^{k,r_1}, \dots, M_{e_t}^{k,r_t}) \geq k\left(1 + \sum_{i=1}^{t}(e_i - 1)\right).$$

Proof. Let $N = k\left(1 + \sum_{i=1}^{t}(e_i - 1)\right) - 1$. Let $L_1, \dots, L_t, R_1, \dots, R_t$ be intervals partitioning $[N]$, with $L_1 = R_1$, such that for $i \in \{1, \dots, t-1\}$, $\max L_{i+1} < \min L_i$ and $\max R_i < \min R_{i+1}$. Further, let $|L_1| = ke_1 - 1$, and for $i \in \{2, \dots, t\}$ let $|L_i| = (k - r_i)(e_i - 1)$ and $|R_i| = r_i(e_i - 1)$. For an edge $X \in \binom{[n]}{k}$, let $c(X) = \max\{i : X \cap (L_i \cup R_i) \neq \emptyset\}$. The interval L_1 is too small for c to contain a copy of $M_{e_1}^{k,r_1}$ in color 1.

Suppose that c contained a copy of $M_{e_i}^{k,r_i}$ in color i for some $i \in \{2, \dots, t\}$. If $r_i = k$, then $L_i = \emptyset$ and $|R_i| = k(e_i - 1)$; therefore some edge of $M_{e_i}^{k,r_i}$ does not intersect R_i and hence does not have color i. The case $r_i = 0$ is similar, except $|L_i| = k(e_i - 1)$ and $R_i = \emptyset$.

Now suppose $1 \leq r_i < k$. Let $p_1, \dots, p_{e_i}$ be the minimum vertices of the edges of $M_{e_i}^{k,r_i}$ and $q_1, \dots, q_{e_i}$ be the set of maximum vertices, hence $p_1 < p_2 < \cdots < p_{e_i} < q_{e_i} < \cdots < q_1$. In fact, $p_m + k - r_i < p_{m+1}$ and $q_m - r_i > q_{m+1}$ for $m = 1, \dots, e_i - 1$. Since each edge receives color i, either $p_m \in L_i$ or $q_m \in R_i$ for all m.

However, because $|L_i| = (k - r_i)(e_i - 1)$ and $|R_i| = r_i(e_i - 1)$, it must be the case that $p_{e_i} \notin L_i$ and $q_{e_i} \notin R_i$. To see this, suppose that $p_{e_i} \in L_i$, then $p_{e_i} - p_1 = (p_{e_i} - p_{e_i - 1}) + \cdots + (p_2 - p_1) > (e_i - 1)(k - r_i)$. This, of course, implies that $p_1 \in L_{i'}$ for some $i' > i$, so the color of edge 1 of the copy of $M_{e_i}^{k,r_i}$ would not receive color i; a contradiction, so $p_{e_i} \notin L_i$. Similarly, $q_{e_i} \notin R_i$.

Thus, the color of edge e_i in the copy of $M_{e_i}^{k,r_i}$ does not receive color i; a contradiction. Therefore, c avoids $M_{e_i}^{k,r_i}$ for all i. □

When all nesting patterns are the same, the bound from Lemma 3.1 is sharp.

Theorem 3.2. *For positive integers $e_1, \dots, e_t$, and $0 \leq r \leq k$,*

$$\mathrm{OR}(M_{e_1}^{k,r}, \dots, M_{e_t}^{k,r}) = k\left(1 + \sum_{i=1}^{t}(e_i - 1)\right).$$

Proof. The lower bound follows from Lemma 3.1. We prove the upper bound by induction on $\sum_{i=1}^{t} e_i$. If $\sum_{i=1}^{t} e_i = t$, then $e_i = 1$ for all i, so $\mathrm{OR}(M_{e_1}^{k,r}, \dots, M_{e_t}^{k,r}) = k$, and the claim holds.

Suppose that $\sum_{i=1}^{t} e_i > t$ and let c be a t-coloring of $E(K_N^k)$ where $N = k\left(1 + \sum_{i=1}^{t}(e_i - 1)\right)$. Suppose that $c(\{1, \dots, r\} \cup \{N - k + r + 1, \dots, N\}) = j$ for some $j \in [t]$. Let G be the graph given by deleting the vertices in $\{1, \dots, r\} \cup \{N - k + r + 1, \dots, N\}$ from K_N^k. Let $e'_j = e_j - 1$ and $e'_i = e_i$ for $i \neq j$. Notice that $G \cong K_{N-k}^k$ and $N - k = k\left(1 + \sum_{i=1}^{t}(e'_i - 1)\right)$. Therefore, since $\sum_{i=1}^{t} e'_i = \sum_{i=1}^{t} e_i - 1$, the induction hypothesis implies that G contains an i-colored copy of $M_{e'_i}^{k,r_i}$ for some i. Since $e'_i = e_i$ when $i \neq j$, we have $i = j$. Then the j-colored copy of $M_{e_j - 1}^{k,r_j}$ along with the edge $\{1, \dots, r\} \cup \{N - k + r + 1, \dots, N\}$ is a j-colored copy of $M_{e_j}^{k,r_j}$. □

Interestingly, as opposed to the large discrepancy between the ordered and ordinary Ramsey numbers of paths, we see that $\mathrm{OR}_t(M_e^{2,r}) \leq 2\,\mathrm{R}_t(M_e^{2,r})$. However, this trend does not continue when the ordering of the matching is not nested as in $M_e^{k,r}$. Likely $M_e^{k,r}$ minimizes the ordered Ramsey number $\mathrm{OR}_t(M)$ among all orderings of k-uniform matchings M on e edges.

Conlon, Fox, Lee and Sudakov [6] explore the ordered Ramsey numbers of 2-uniform matchings.

Theorem 3.3 (Conlon, Fox, Lee and Sudakov [6]). *Let $M_2, \ldots, M_t$ be ordered 2-uniform matchings, and let $p \geq 2$. Then $\mathrm{OR}(K_p, M_2, \ldots, M_t) \leq \mathrm{OR}(M_2, \ldots, M_t)^{\lceil \lg p \rceil}$. Therefore, for an ordered 2-uniform matching M with e edges, $\mathrm{OR}_t(M) \leq (2e)^{\lceil \lg(2e) \rceil^{t-1}} \leq 2^{\lceil \lg(2e) \rceil^t}$.*

Compare the upper bound here with the lower bound from Theorem 2.2, showing that this upper bound is nearly tight. In terms of e, the bound above is quasi-polynomial, but in terms of t the bound is doubly-exponential.

Define the k-uniform graph G_s^k iteratively on s as follows: let G_0^k consist of a single vertex, and for $s \geq 1$, let G_s^k consist of k disjoint, consecutive copies of G_{s-1}^k, and introduce every k-uniform edge consisting of exactly one vertex from each copy. Notice that $G_s^2 = K_{2^s}$.

Using the graph G_s^k, we attain a bound on the t-color ordered Ramsey numbers of certain "nice" orderings of k-uniform matchings. This bound is a generalization of Theorem 3.3, where G_s^k replaces the complete graph.

Lemma 3.4. *Let $M_2, \ldots, M_t$ be any k-uniform ordered matchings and $s \geq 0$. Then*

$$\mathrm{OR}(G_s^k, M_2, \ldots, M_t) \leq \mathrm{OR}(M_2, \ldots, M_t)^s.$$

Proof. We prove by induction on s. When $s = 0$, the graph G_0^k consists of a single vertex, and hence every coloring of K_1^k contains a copy of G_s^k in every color.

Suppose that $s > 0$ and let $r = \mathrm{OR}(M_2, \ldots, M_t)$. Suppose, for the sake of contradiction, that c is a t-coloring of $K_{r^s}^k$ that avoids a j-colored copy of M_j for each $j \in \{2, \ldots, t\}$ and avoids a 1-colored copy of G_s^k. Let $V_1, \ldots, V_r$ be equal-sized intervals partitioning $[r^s]$ such that $\max V_i < \min V_{i+1}$ for $i \in [r-1]$. By the induction hypothesis, restricting c to V_i yields either a copy of G_{s-1}^k in color 1 or a j-colored copy of M_j for some $j \in \{2, \ldots, t\}$. Since c contains no j-colored copy of M_j, each V_i contains a copy of $G_{(s-1)}^k$. Since c avoids G_s^k, then for any indices $1 \leq i_1 < \cdots < i_k \leq r$ there must be $x_{i_j} \in V_{i_j}$ such that $c(x_{i_1}, \ldots, x_{i_k}) \neq 1$. Define a coloring of $E(K_r^k)$ by letting $c'(v_{i_1}, \ldots, v_{i_k})$ be any color in $\{c(x_{i_1}, \ldots, x_{i_k}) : x_{i_j} \in V_{i_j}\} \setminus \{1\}$. By the definition of r, c' contains an j-colored copy of M_j for some $j \in \{2, \ldots, t\}$ and therefore c also contains a j-colored copy of M_j; a contradiction. $\square$

Let M be an ordered k-uniform matching on vertex set $[ke]$. We say that M is *k-nestable* if there exist disjoint intervals $I_1, \ldots, I_k$, some of which may be empty or degenerate, spanning $[ke]$ such that $1 \in I_1, ke \in I_k$, where each edge in M either is contained in some interval I_j or intersects all intervals $I_1, \ldots, I_k$, and for each $j \in [k]$ the edges contained within I_j form a matching, denoted M_j, that is either k-nestable or empty. A set of intervals $I_1, \ldots, I_k$ satisfying these properties is a *k-nesting* of M. Notice that every matching contained as a subgraph of G_s^k for some s must be k-nestable; in particular, every 2-uniform matching is 2-nestable as $G_s^2 \cong K_{2^s}$. The following lemma provides the converse to this observation.

Lemma 3.5. *If M is a k-uniform hypergraph consisting of a k-nestable matching on e edges and v additional isolated vertices, then M can be embedded into $G_{e+\lceil \log_k(e+v) \rceil}^k$.*

Proof. We prove by induction on e. If $e = 0$, then the claim holds immediately through the fact that G_s^k has k^s vertices.

Now suppose that $e \geq 1$. Let $I_1, \ldots, I_k$ be a k-nesting of M and let M_j be graph with vertex set I_j and edge set $E(M) \cap \binom{I_j}{k}$. Also let $M' = M - \bigcup_j M_j$. In other words, M_j is the matching induced on interval I_j along with all other vertices contained in I_j, and M' is the set of edges that intersect every interval. Notice that some of the M_j's may be empty or only consist of isolated vertices and that M' may be empty as well. Let $e' = |E(M')|$, $e_j = |E(M_j)|$ and v_j be the the number of isolated vertices of M_j.

Let $r = \max_j(e_j + \lceil \log_k(e_j + v_j) \rceil)$, then because $e_j < e$ for all j, M_j can be embedded into G_r^k by the inductive hypothesis. Thus, by embedding M_j into the j'th copy of G_r^k in G_{r+1}^k, we attain an embedding of $\bigcup_j M_j$ into G_{r+1}^k. Finally, it is easy to add the edges of M' into this embedding because the j'th vertex in an edge of M' has been embedded into the j'th copy of G_r^k in G_{r+1}^k due to the original k-nesting of M. Hence, we have an embedding of M into G_{r+1}^k.

Notice that $e_j \leq \min\{e - e', e - 1\}$ for all j and that $v_j \leq v + e'$ because e' new isolated vertices were added to each interval upon ignoring the edges of M'. Therefore, $e_j + 1 \leq e$ and $e_j + v_j \leq e + v$, so $r + 1 \leq e + \lceil \log_k(e + v) \rceil$. We conclude that M embeds into $G_{e+\lceil \log_k(e+v) \rceil}^k$. □

Notice that, Lemma 3.5 implies that a k-nestable matching on e edges embeds into $G_{e+\lceil \log_k e \rceil}^k$. In many cases, Lemma 3.5 will not be tight as $\max_j(e_j + \lceil \log_k(e_j + v_j) \rceil)$ may be substantially smaller than $e + \lceil \log_k(e + v) \rceil$; however, there are k-nestable matchings which come close to showing the tightness of the lemma. It is easy to observe that for $1 \leq r \leq k - 1$, $M_e^{k,r}$ embeds into G_{e+1}^k but not into G_e^k whenever $e \geq 2$. Thus, if $e \leq k$, $M_e^{k,r}$ embeds into $G_{e+\lceil \log_k e \rceil}^k$ but not into G_s^k for any $s < e + \lceil \log_k e \rceil$.

The following theorem follows from Lemmas 3.4 and 3.5 and the fact that $\mathrm{OR}_1(M) = ek$ if M is a k-uniform ordered matching with e edges.

Theorem 3.6. *Let $k \geq 3$ and $e \geq 2$. If M is a k-nestable ordered matching with e edges, then* $\mathrm{OR}_t(M) \leq (ek)^{\lceil e+\log_k e \rceil^{t-1}} = k^{\lceil e+\log_k e \rceil^{t-1}(1+\log_k e)}$.

This extends the previous bound on 2-uniform matchings [6]. While the bound remains doubly-exponential in terms of t, the bound has increased from quasi-polynomial to exponential in terms of e.

Notice that for these "nice" orderings of a k-uniform matching on e edges, the bound on the ordered Ramsey number $\mathrm{OR}_t(M)$ is only slightly larger than the ordered Ramsey number $\mathrm{OR}_t(P_e^{k,\ell})$ of the naturally-ordered (k, ℓ)-path on e edges when $i(k, \ell) = 3$.

We say that a k-uniform ordered matching M is *simply interlacing* if for any pair of distinct edges A, B in M, where $A = \{a_1 < a_2 < \cdots < a_k\}$ and $B = \{b_1 < b_2 < \cdots < b_k\}$ either a_i and b_i are consecutive in $A \cup B$ for each i or there is some i where $a_i < b_1 < b_k < a_{i+1}$ (where $a_0 = -\infty$ and $a_{k+1} = +\infty$). If the former holds, we say that A and B *interlace*, and if the latter holds, we say that A and B *nest*. Notice that every 2-uniform matching is simply interlacing.

Corollary 3.7. *If $k \geq 3$, $e \geq 2$, and M is a simply-interlacing k-uniform ordered matching with e edges, then M is k-nestable; hence* $\mathrm{OR}_t(M) \leq k^{\lceil e+\log_k e \rceil^{t-1}(1+\log_k e)}$.

Proof. By Theorem 3.6, it suffices to show that M is k-nestable. Define a relation on the edges of M by $A \preceq B$ if $A = B$ or if $b_i < a_1 < a_k < b_{i+1}$ for some $0 \leq i \leq k-1$, where $A = \{a_1 < \cdots < a_k\}$ and $B = \{b_1 < \cdots < b_k\}$ (again under the convention that $b_0 = -\infty$). We observe that $\preceq$ is not quite a partial ordering. Suppose that $A = \{a_1, \ldots, a_k\}$, $B = \{b_1, \ldots, b_k\}$ and $C = \{c_1, \ldots, c_k\}$ where $a_k < b_1$, $b_k < c_k$, and $a_i < c_i < a_{i+1}$ for all $1 \leq i \leq k-1$. Thus, $A \preceq B$ and $B \preceq C$, but $A \not\preceq C$. Thus, $\preceq$ is not a transitive relation. However, $\preceq$ is reflexive and antisymmetric, so $\preceq$ admits "maximal" elements in the sense that A is maximal if there is no $B \neq A$ such that $A \preceq B$. Let $A_1, \ldots, A_p$ be the edges of M that are either maximal with respect to $\preceq$ or interlace with some maximal edge. Therefore, it must be the case that A_i and $A_{i'}$ interlace. We refer to these edges as *spanning edges*.

For each $i \in [p]$, label the vertices in A_i as $A_i = \{a_{i,1} < \cdots < a_{i,k}\}$; also let $a_{i,0} = -\infty$ and $a_{i,k+1} = +\infty$. Observe that for each $j \in [k-1]$, we have $\max_{i\in[p]} a_{i,j} < \min_{i\in[p]} a_{i,j+1}$, as otherwise there is a pair of edge A_i and $A_{i'}$ where $a_{i,j} > a_{i',j+1}$ and hence $a_{i,j}$ and $a_{i',j}$ are not consecutive in $A_i \cup$

$A_{i'}$. Therefore, we can define disjoint intervals $I_1, \dots, I_k$ such that $I_j = [\min_{i\in[p]} a_{i,j}, \max_{i\in[p]} a_{i,j}]$. These intervals do not necessarily span $V(M)$, but we will expand them to include vertices not in $A_1, \dots, A_p$.

For a non-spanning edge B in M, there is at least one edge A_i where $B \prec A_i$. Therefore, there exists a $j \in \{0, \dots, k-1\}$ such that $a_{i,j} < \min B < \max B < a_{i,j+1}$. Observe that since $k \geq 3$, for any $i' \in [p]$ the edge B is comparable to $A_{i'}$ since there is some $a_{i',j'}$ not in the interval $[a_{i,j}, a_{i,j+1}]$. While it may not be the case that $B \prec A_{i'}$, it is true that for every $i' \in [p]$ and $a_{i',j+c_{i'}} < \min B < \max B < a_{i',j+c_{i'}+1}$ for some $c_{i'} \in \{-1, 0, +1\}$, as $A_{i'} \prec B$ only when $a_{i',k} < \min B$. Therefore, let j_B be the minimum integer satisfying $j_B \geq 1$ and $j_B \geq j + c_{i'}$ for each $i' \in [p]$.

If B, B' are two non-spanning edges in M and $j_B < j_{B'}$, then $\max B < a_{i,j_B+1}$ for all $i \in [p]$ and $a_{i',j_{B'}} < \min B'$ for some $i' \in [p]$. Then $\max B < a_{i',j_B+1} < \min B'$. Therefore, if for every non-spanning edge B in M we minimally extend the interval I_{j_B} to contain the edge B, the intervals $I_1, \dots, I_k$ will always be disjoint.

Note that the matching M_j given by the edges entirely within the interval I_j is a simply-interlacing k-uniform ordered matching and hence is k-nestable by an inductive argument. Therefore, the intervals $I_1, \dots, I_k$ form a k-nesting of M. □

We conclude by noting that Lemma 3.4 will not apply to most ordered k-uniform matchings for $k \geq 3$. For $k \geq 4$, let A and B be defined as

$$A = \{1, \dots, \lfloor k/2 \rfloor\} \cup \{k+1, \dots, k + \lceil k/2 \rceil\}, \quad B = \{\lfloor k/2 \rfloor + 1, \dots, k\} \cup \{k + \lceil k/2 \rceil, \dots, 2k\}.$$

Observe that the ordered matching with edges A and B is not k-nestable. While every ordered 3-uniform matching on two edges is 3-nestable, there exists an ordered 3-uniform matching that is not 3-nestable. A randomly-ordered matching contains these configurations with high probability, so the bound of Theorem 3.6 does not apply to most ordered matchings.

4. Future Directions

Our investigation into k-uniform matchings provides upper bounds that are similar to the previous bounds in the 2-uniform case. Extending the techniques from 2-uniform matchings comes at the cost that it does not apply to all k-uniform ordered matchings, but they do provide bounds that are exponential and not a tower. However, our methods do not allude to lower bounds, and hence it is unclear whether our upper bounds are tight.

The largest question left open from our study of ordered Ramsey numbers is related to arbitrary orderings of (k,ℓ)-paths. While we found upper bounds on $\mathrm{OR}_t(P_e^{2,1})$, our techniques did not easily extend to higher uniformities. Upper bounds on $\mathrm{OR}_t(P_e^{k,\ell})$ for arbitrary orderings of $P_e^{k,\ell}$ would be very interesting and would significantly extend our current techniques. Noticing that $\mathrm{tow}_{k-2}(\Omega(n^2)) \leq \mathrm{R}_2(K_n^k) \leq \mathrm{tow}_{k-1}(O(n))$ (see [7]), the bound for $\mathrm{OR}_t(P_e^{k,k-1})$ for the natural ordering cannot be far off a general bound for $\mathrm{OR}_t(P_e^{k,k-1})$ for an arbitrary ordering. However, $\mathrm{OR}_t(P_e^{k,\ell})$ for the natural ordering grows as a tower of height $i(k,\ell) - 2$ in terms of e, so the upper bound for $\mathrm{OR}_t(P_e^{k,\ell})$ for an arbitrary ordering may be much larger, especially if $i(k,\ell) = 2$. Thus, bounds on tight paths may not lead to bounds on loose paths in the same way that Theorem 1.1 draws this connection for monotone paths.

The generalized diamond D_r consists of r copies of $P_2^{2,1}$ who share first and last vertices. The ordering of the intermediate vertices is unimportant as all orderings yield isomorphic graphs. Balko, Cibulka, Král, and Kynčl [1] determined that $\mathrm{OR}_2(D_2) = 11$. We would like to determine, asymptotically or otherwise, the growth of $\mathrm{OR}_t(D_r)$ in terms of r. While the study of monotone paths explains what happens when a graph gets “longer,” the study of the generalized diamond will yield a better understanding of what happens when a graph gets “wider.”

The natural extension of D_r to higher uniformities, $D_r^{k,\ell}$, consists of r copies of $P_2^{k,\ell}$ who share their first $k-\ell$ and last $k-\ell$ vertices. However, unless $\ell = 1$, $D_r^{k,\ell}$ admits many nonisomorphic orderings of the intermediate vertices, none of which are essentially natural. Presumably, a somewhat symmetric ordering of the intermediate vertices will minimize $\mathrm{OR}_t(D_r^{k,\ell})$, but other than the fact that it is bounded below by $\mathrm{OR}_t(P_2^{k,\ell})$, it is unclear how large this number can become.

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