RAMSEY NUMBERS FOR PARTIALLY-ORDERED SETS

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Abstract. We present a refinement of Ramsey numbers by considering graphs with a partial ordering on their vertices. This is a natural extension of the ordered Ramsey numbers. We formalize situations in which we can use arbitrary families of partially-ordered sets to form host graphs for Ramsey problems. We explore connections to well studied Turán-type problems in partially-ordered sets, particularly those in the Boolean lattice. We find a strong difference between Ramsey numbers on the Boolean lattice and ordered Ramsey numbers when the partial ordering on the graphs have large antichains.

1. Introduction

Ramsey and Turán problems are fundamental to graph theory. Turán problems focus on the maximum size of objects that forbid a certain substructure whereas Ramsey problems concern partitioning an object into parts where each part forbids a certain substructure. Traditionally, these problems are considered in the domain of graphs. Recently, Ramsey problems have been extended to graphs with a total ordering on their vertices [2, 4, 5, 7, 8, 13, 25, 26], and Turán problems have been considered within the Boolean lattice [9, 10, 16, 17, 18, 19, 24]. We unite and generalize these concepts into Ramsey theory on partially-ordered sets.

Ramsey numbers describe the transition where it becomes impossible to partition a complete graph into t parts such that each part does not contain a certain subgraph. For k-uniform hypergraphs $G_1, \dots, G_t$, the *t-color graph Ramsey number* $\mathrm{R}^k(G_1, \dots, G_t)$ is the least integer N such that any t-coloring of the edges of the k-uniform complete graph on N vertices contains a copy of G_i in color i for some $i \in \{1, \dots, t\}$; when $G_1 = \cdots G_t = G$, we shorten the notation to $\mathrm{R}^k_t(G)$. Since $\mathrm{R}^k_t(K_n)$ is finite for all t and n, all Ramsey numbers exist, including the generalizations we discuss in this paper. In our notation for Ramsey numbers, we use k to emphasize that $G_1, \dots, G_t$ are k-uniform graphs.

A k-uniform *ordered hypergraph* is a k-uniform hypergraph G with a total order on the vertex set $V(G)$. An ordered hypergraph G *contains* another ordered hypergraph H exactly when there exists an embedding of H in G that preserves the vertex order. For ordered k-uniform hypergraphs $G_1, \dots, G_t$, the *ordered Ramsey number* $\mathrm{OR}^k(G_1, \dots, G_t)$ is the least integer N such that every t-coloring of the edges of the complete k-uniform graph with vertex set $\{1, \dots, N\}$ contains an ordered copy of G_i in color i for some $i \in \{1, \dots, t\}$. Since there is essentially one ordering of the complete graph, $\mathrm{OR}^k(G_1, \dots, G_t) \le \mathrm{R}^k_t(K_n)$ for $n = \max\{|V(G_i)| : i \in \{1, \dots, t\}\}$. In general, $\mathrm{OR}^k_t(G)$ can be much larger than $\mathrm{R}^k_t(G)$, such as when G is an ordered path. Ordered Ramsey numbers on ordered paths have deep connections to the Erdős-Szekeres Theorem and the Happy Ending Problem [12] (see [13, 26]).

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A *partially-ordered set*, or *poset*, is a pair $(X, \leq)$ where X is a set and $\leq$ is a relation such that $\leq$ is reflexive, anti-symmetric, and transitive. A pair $x, y \in X$ is *comparable* if $x \leq y$ or $y \leq x$, and a k*-chain* is a set of k distinct, pairwise comparable elements. A k-uniform *partially-ordered hypergraph*, or *pograph*, is a k-uniform hypergraph H and a relation $\leq$ such that $(V(H), \leq)$ is a poset and every edge in the edge set $E(H)$ is a k-chain in $(V(H), \leq)$; note that it is not necessary that every k-chain be an edge. If G and H are k-uniform pographs, then G *contains* H if there is an injection from $V(H)$ to $V(G)$ that preserves comparisons in H and sends edges of H to edges of G. Let $\mathcal{P} = \{P_n : n \geq 1\}$ be a family of posets such that $P_n \subseteq P_{n+1}$ for each n and let $H_1, \ldots, H_t$ be k-uniform pographs. The *partially-ordered Ramsey number* $\mathrm{R}^k_{\mathcal{P}}(H_1, \ldots, H_t)$ is the minimum N such that every t-coloring of the k-chains of P_N contains a copy of H_i in color i for some $i \in \{1, \ldots, t\}$. The pographs $H_1, \ldots, H_t$ are contained within ordered hypergraphs $G_1, \ldots, G_t$ by extending the partial order to a total order; if P_n contains a chain of size $\mathrm{OR}^k(G_1, \ldots, G_t)$, then $\mathrm{R}^k_{\mathcal{P}}(H_1, \ldots, H_t) \leq n$. Thus, partially-ordered Ramsey numbers exist whenever the family $\mathcal{P}$ has unbounded height. This is not a requirement, and we discuss several interesting poset families and their relations to other Ramsey numbers in Section 5. For the majority of this paper, we will focus on two natural poset families and use special notation to describe their Ramsey numbers. Let $H_1, \ldots, H_t$ be k-uniform pographs.

(1) Let C_n be a chain of n elements, $\mathcal{C} = \{C_n : n \geq 1\}$, and define the *chain Ramsey number* $\mathrm{CR}^k(H_1, \ldots, H_t) = \mathrm{R}^k_{\mathcal{C}}(H_1, \ldots, H_t)$.
(2) Let B_n be the Boolean lattice of subsets of $\{1, \ldots, n\}$, $\mathcal{B} = \{B_n : n \geq 1\}$, and define the *Boolean Ramsey number* $\mathrm{BR}^k(H_1, \ldots, H_t) = \mathrm{R}^k_{\mathcal{B}}(H_1, \ldots, H_t)$.

When $H_1 = \cdots = H_t = H$, we shorten our notation to $\mathrm{CR}^k_t(H) = \mathrm{CR}^k(H_1, \ldots, H_t)$ and $\mathrm{BR}^k_t(H) = \mathrm{BR}^k(H_1, \ldots, H_t)$. The 2-uniform chain Ramsey numbers are a slight generalization of both ordered Ramsey numbers (if $H_1, \ldots, H_t$ are totally ordered) and the *directed Ramsey numbers*[1] defined by Choudum and Ponnusammy [4], which consider coloring the edges of the transitive tournament to avoid monochromatic copies of certain directed acyclic graphs.

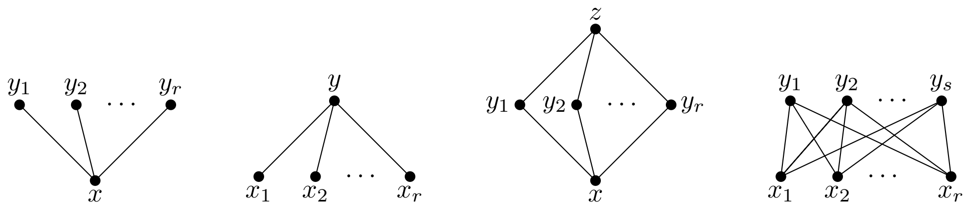
We mainly focus on 1- and 2-uniform Boolean Ramsey numbers, generalizing to other families only when the proof method is identical. The 2-uniform Boolean Ramsey numbers are an interesting generalization of 2-uniform ordered Ramsey numbers, and we discuss them in Section 2. There is little interest in 1-uniform chain Ramsey numbers of graphs as they can be determined by basic application of the pigeonhole principle. The 1-uniform Boolean Ramsey numbers relate to the very active area of 1-uniform Turán problems in the Boolean lattice [9, 10, 16, 17, 18, 19, 24]. This area dates back to Sperner [27] who showed that the largest family of B_n that does not contain a comparable pair has size $\binom{n}{\lfloor n/2 \rfloor}$. These problems ask for the largest collection of elements in the Boolean lattice whose induced subposet does not contain a copy of a specific poset P. Gunderson, Rödl and Sidorenko [20] and Johnston, Lu and Milans [23] considered a Ramsey-type number for the 1-uniform case, but required the copies to be *induced*, which is a stronger condition than our definition. We discuss 1-uniform Boolean Ramsey numbers in Section 3.

Finding an exact value of a Boolean Ramsey number is very difficult. We discuss computational methods to find small Boolean Ramsey numbers in Section 4.

1.1. **Notation and Common Posets.**

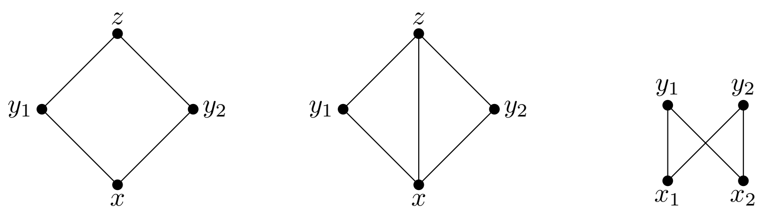
We follow standard notation from [29]. For integers $m \leq n$, we let $[n] = \{1, \ldots, n\}$ and $[m, n] = \{m, m+1, \ldots, n-1, n\}$. We use $\lg n = \log_2 n$ for shorthand.

[1] In [4] these are called ordered Ramsey numbers. See [25] for a detailed discussion about the distinction.



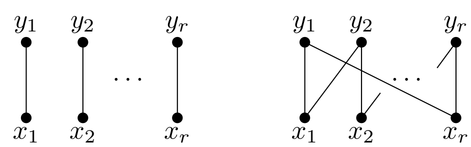
(a) The r-cup, $\vee_r$. (b) The r-cap, $\wedge_r$. (c) The r-diamond, $\Diamond_r$. (d) The r,s-butterfly, $\bowtie_r^s$

FIGURE 1.1. The cup, cap, diamond, and butterfly pographs, respectively.



(a) The diamond, $\Diamond = \Diamond_2$. (b) The 2-dimensional Boolean lattice, B_2. (c) The butterfly, $\bowtie = \bowtie_2^2$

FIGURE 1.2. The 2-diamond, B_2, and $\bowtie$.



(a) The r-matching, M_r. (b) The r-crown, W_r.

FIGURE 1.3. The matching and the crown.

For a poset P and an element $x \in P$, we use $\mathcal{D}(x) = \{y \in P : y \leq x\}$ and $\mathcal{U}(x) = \{y \in P : x \leq y\}$, called the *down-set of* x and the *up-set of* x respectively. For a t-coloring c of the 2-chains in P, an element $x \in P$, and a color $i \in [t]$, we define the *i-colored down-set of* x, denoted $\mathcal{D}_i(x)$, to be the elements $y < x$ such that $c(yx) = i$; similarly the *i-colored up-set of* x, denoted $\mathcal{U}_i(x)$, is the set of elements $y > x$ such that $c(xy) = i$. The *height* of a poset P, denoted $h(P)$, is the maximum size of a chain in P.

When we discuss 1-uniform pographs, we define only the poset and assume the set of "edges" is the same as the set of elements. In the case of 2-uniform pographs, we have two natural options for the edge set. For a poset $(P, \leq)$, the *comparability graph* is the pograph with vertex set P and an edge uv if and only if $u < v$. The *Hasse diagram* of $(P, \leq)$ is the pograph with vertex set P and an edge uv if and only if $u < v$ and there does not exist an element w such that $u < w$ and $w < v$; such pairs uv are *cover relations*. When we draw a pograph, adjacent vertices are comparable with the comparison ordered by height. We will focus mainly on a few natural 2-uniform pographs.

- The *n-chain*, denoted C_n, is the Hasse diagram of n totally-ordered elements.
- The *n-dimensional Boolean lattice*, denoted B_n, is the comparability graph of subsets of $[n]$ ordered by subset inclusion. See Figure 2(b) for a diagram of B_2.

- The *r-cup*, denoted $\vee_r$, is the comparability graph of the poset with elements $\{x, y_1, \ldots, y_r\}$ where $x \leq y_i$ for all i (see Figure 1(a)).
- The *r-cap*, denoted $\wedge_r$, is the comparability graph of the poset with elements $\{y, x_1, \ldots, x_r\}$ where $x_i \leq y$ for all i (see Figure 1(b)).
- The *r-diamond*, denoted $\Diamond_r$, is the Hasse diagram of the poset with elements $\{x, y_1, \ldots, y_r, z\}$ where $x \leq y_i \leq z$ for all i (see Figures 1(c) and 2(a)).
- The *r, s-butterfly*, denoted $\bowtie_r^s$, is the comparability graph of the poset with elements $\{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_s\}$ where $x_i \leq y_j$ for all i and j; we use $\bowtie$ to denote $\bowtie_2^2$ (see Figures 1(d) and 2(c)).
- The *matching of size* n, denoted M_n, is the comparability graph of the poset with elements $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ where $x_i \leq y_i$ for all i (Figure 3(a)).
- The *crown graph of order* n, denoted W_n, is the comparability graph of the poset with elements $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ where $x_i \leq y_i$ and $x_i \leq y_{i+1 \,(\mathrm{mod}\; n)}$ for all i (Figure 3(b)).

Note the difference between the 2-diamond $\Diamond_2$ and the 2-dimensional Boolean lattice B_2. Both pographs are defined for the same poset, but $\Diamond_2$ is the Hasse diagram and hence has one fewer edge than the comparability graph in B_2. This distinction leads to different values of 2-uniform Boolean Ramsey numbers; see Section 4.

Additionally, keep in mind that we often use the same symbol to denote both the 1- and 2-uniform pographs associated with a given poset. Whether we are discussing the 1- or 2-uniform case will always be clear from context.

2. 1-Uniform Boolean Ramsey Numbers

For a poset P, define $e(P)$ to be the maximum m such that, for all n, the union of the middle m levels of B_n does not contain a copy of P. The parameter $e(P)$ is very common in the study of Turán-type problems in posets.

Proposition 2.1. Let $P_1, \ldots, P_t$ be posets. If M is the least integer such that $P_i \subseteq B_M$ for all i, then

$$\max\left\{M, \sum_{i=1}^{t} e(P_i)\right\} \leq \mathrm{BR}^1(P_1, \ldots, P_t) \leq \sum_{i=1}^{t}(|P_i| - 1).$$

Proof. The upper bound follows from the fact that B_n contains a chain of length $n+1$ and that $P_i \subseteq C_{|P_i|}$. For the lower bound, let $n = \sum_{i=1}^{t} e(P_i) - 1$. For $v \in B_n$ with $|v| \in \left[\sum_{j=1}^{i-1} e(P_j), \sum_{j=1}^{i} e(P_j) - 1\right]$, let $c(v) = i$. Thus $c^{-1}(i)$ is the union of $e(P_i)$ consecutive levels of B_n, so c avoids copies of P_i in color i for all i. □

Later in this section, we will demonstrate situations in which the lower bound in Proposition 2.1 is not tight, but we believe that the lower bound is *essetially* correct in a sense that we will make exact later. To this end, we believe that the upper bound in Proposition 2.1 is far from tight in most cases and conjecture that the upper bound is tight *only* when each P_i is a chain (and hence $e(P_i) = |P_i| - 1$).

The remainder of this section sets out to determine the 1-uniform Boolean Ramsey numbers of various posets.

Theorem 2.2. *For positive integers* $n_1, \ldots, n_t$, $\mathrm{BR}^1(B_{n_1}, C_{n_2}, \ldots, C_{n_t}) = n_1 + \sum_{i=2}^{t}(n_i - 1)$.

Proof. The lower bound follows from Proposition 2.1, so we need only show the upper bound.

We first prove that $\mathrm{BR}^1(B_n, C_m) \le n+m-1$ by induction on m. For $m = 1$, the result is immediate as any use of color 2 creates a chain of order 1. Suppose that $m \ge 2$ and let $N = n+m-1$. Let c be any 2-coloring of B_N and suppose that c avoids copies of C_m in color 2; we will show that c must admit a copy of B_n in color 1. Let L be the family of subsets of $[N-1]$. As L is a copy of B_{N-1}, the induction hypothesis states that c restricted to L must admit either a copy of B_n in color 1 or a copy of C_{m-1} in color 2. If the former holds, then we are done. Otherwise, c restricted to L admits a copy of C_{m-1} in color 2. Suppose that $X_1, \ldots, X_s$ are the copies of C_{m-1} in color 2 contained in L. Because c avoids copies of C_m in color 2, we see that the elements in $\bigcup_{i=1}^s (\mathcal{U}(\max X_i) \setminus \max X_i)$ all have color 1. Let $U = \bigcup_{i=1}^s \mathcal{U}(\max X_i) \cap L$ and let $U' = \{Y \cup \{N\} : Y \in U\}$. Notice that $U' \subseteq \bigcup_{i=1}^s (\mathcal{U}(\max X_i) \setminus \max X_i)$, so U' contains only elements of color 1. Furthermore, it is easily seen that B_{N-1} embeds into $(L \setminus U) \cup U'$ as $U \cong U'$ and U' is an up-set. However, c restricted to $(L \setminus U) \cup U'$ does not contain any copies of C_{m-1} in color 2, so by the induction hypothesis, it must admit a copy of B_n in color 1 as needed. We conclude that $\mathrm{BR}^1(B_n, C_m) \le N$.

Now that we have proved that $\mathrm{BR}^1(B_n, C_m) = n+m-1$, the t-color version follows quickly. Let $m = 1 + \sum_{i=2}^t (n_i - 1)$ and $N = n_1 + m - 1$.. From the 2-color case, $\mathrm{BR}^1(B_{n_1}, C_m) = N = n_1 + \sum_{i=2}^t (n_i - 1)$. Thus, if c is a t-coloring of B_N, then either c admits a copy of B_{n_1} in color 1 or there exists a copy of C_m where all elements have color in $\{2, \ldots, t\}$. If there is a copy of B_{n_1} in color 1, then we are done. If not, since $m = 1 + \sum_{i=2}^t (n_i - 1)$, there exists a chain of size n_i within the copy of C_m that has color i for some $i \in \{2, \ldots, t\}$ by the pigeonhole principle. □

A common tool in studying Turán-type questions in posets is known as the *Lubell function*. For a family $\mathcal{F} \subseteq B_n$, the *Lubell function of* $\mathcal{F}$ is defined as

$$\mathrm{lu}_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1}.$$

The Lubell function of $\mathcal{F}$ can be interpreted as the average size of $|\mathcal{F} \cap \mathcal{C}|$ where $\mathcal{C}$ is a full chain in B_n. An alternate interpretation is that $\mathrm{lu}_n(\mathcal{F})$ is the expected number of elements of $\mathcal{F}$ that are visited by a random walk from the empty set to the full set along the Hasse diagram of B_n. Using either interpretation, it is straightforward to observe that $|\mathcal{F}| \le \mathrm{lu}_n(\mathcal{F})\binom{n}{\lfloor n/2 \rfloor}$. It is due to this observation that Lubell functions of P-free families have received a great deal of attention, as bounds on the Lubell function help answer Turán-type questions in the Boolean lattice. We apply Lubell functions to attain bounds on the 1-uniform Boolean Ramsey number by calling upon linearity, i.e. if $\mathcal{F} \cap \mathcal{G} = \varnothing$, then $\mathrm{lu}_n(\mathcal{F} \cup \mathcal{G}) = \mathrm{lu}_n(\mathcal{F}) + \mathrm{lu}_n(\mathcal{G})$.

Johnston, Lu and Milans [23] use Lubell functions to explore a question similar to the 1-uniform Boolean Ramsey number. For two posets P and Q, we say that Q contains an *induced* copy of P if there is an injection $\phi : P \to Q$ where $x \le_P y$ if and only if $\phi(x) \le_Q \phi(y)$. Johnston, Lu and Milans show that if $N \ge (2t)^{2^{n-1}} + 1$, then any t-coloring of B_N contains a monochromatic *induced* copy of B_n. This implies that

$$\mathrm{BR}_t^1(B_n) \le (2t)^{2^{n-1}} + 1.$$

The trivial upper bound in Proposition 2.1 shows that $\mathrm{BR}_t^1(B_n) \le t(2^n - 1)$, so the bound attained through looking for *induced* copies of B_n does not provide a good estimate of $\mathrm{BR}_t^1(B_n)$. Despite this, we will make use of the technique employed by Johnston, Lu and Milans to prove their doubly exponential bound. Lemma 2.3 should be seen as a formalization of this stepping stone. For a poset P, let $L_n(P)$ be the maximum value $\mathrm{lu}_n(\mathcal{F})$ among families $\mathcal{F} \subseteq B_n$ such that $\mathcal{F}$ is P-free.

Lemma 2.3. *If $P_1, \ldots, P_t$ are posets and $\sum_{i=1}^t L_n(P_i) < n+1$ for some integer n, then $\mathrm{BR}^1(P_1, \ldots, P_t) \le n$.*

Proof. Let n be such that $\sum_{i=1}^{t} L_n(P_i) < n+1$. Let c be any t-coloring of B_n and for $i \in [t]$, let $\mathcal{F}_i = c^{-1}(i)$. By linearity,

$$n+1 = \mathrm{lu}_n(B_n) = \mathrm{lu}_n\left(\bigcup_{i=1}^{t} \mathcal{F}_i\right) = \sum_{i=1}^{t} \mathrm{lu}_n(\mathcal{F}_i).$$

As $\sum_{i=1}^{t} L_n(P_i) < n+1$, there is some $i \in [t]$ for which $\mathrm{lu}_n(\mathcal{F}_i) > L_n(P_i)$. Hence, $\mathcal{F}_i$ is not P_i-free, so c admits a copy of P_i in color i, and $\mathrm{BR}^1(P_1, \ldots, P_t) \le n$. □

Griggs and Li [17] define a poset P to be *uniformly Lubell-bounded*, or *uniformly L-bounded*, if $L_n(P) \le e(P)$ for all n. By a direct application of Lemma 2.3, we find that the lower bound given in Proposition 2.1 is tight when regarding uniformly L-bounded posets.

Proposition 2.4. If $P_1, \ldots, P_t$ are uniformly L-bounded posets, then

$$\mathrm{BR}^1(P_1, \ldots, P_t) = \sum_{i=1}^{t} e(P_i).$$

Up until this point, we have mostly considered cases in which the lower bound in Proposition 2.1 is tight. This, however, is not the case in general. To show this, we consider the *butterfly poset*. The butterfly poset is special as De Bonis, Katona, and Swanepool [10] determined the largest $\bowtie$-free family in B_n to be *exactly* the middle two levels *for all* n, while most other results in this direction are necessarily asymptotic. This is especially interesting as $L_n(\bowtie) = 3$ for all n, which is witnessed by any family consisting of a level of B_n along with $\varnothing$ and $[n]$. As such, Lemma 2.3 implies that $\mathrm{BR}_t(\bowtie) \le 3t$, but this is not tight. In order to determine $\mathrm{BR}_t^1(\bowtie)$, we require a more careful use of the idea in Lemma 2.3. For a poset P, define

$$L_n'(P) = \max\left\{\mathrm{lu}_n(\mathcal{F}) : \mathcal{F} \subseteq B_n \setminus \{[n], \varnothing\}, \mathcal{F} \text{ is } P\text{-free}\right\},$$

This new value $L_n'(P)$ is the maximum Lubell value of a P-free family that does not contain either the maximal or minimal element.

Proposition 2.5. For $t \ge 1$, $\mathrm{BR}_t^1(\bowtie) = 2t+1$.

Proof. Lower bound. Let c be a t-coloring of B_{2t} defined as follows. For $i \in [t-1]$, if $|x| \in \{2i, 2i+1\}$, let $c(x) = i$, and if $|x| \in \{0, 1, 2t\}$, let $c(x) = t$. As $e(\bowtie) = 2$, we see that c avoids copies of $\bowtie$ in colors $1, \ldots, t-1$. Further, it is easy to check that $\bowtie$ does not appear in color t, so $\mathrm{BR}_t^1(\bowtie) > 2t$.

Upper bound. As shown by Griggs and Li [16, Theorem 5.1], $L_n'(\bowtie) = 2$. We begin by showing that for any n, $L_n'(\vee_2) < 2$. Suppose that $L_n'(\vee_2) \ge 2$ and let $\mathcal{F} \subseteq B_n \setminus \{[n], \varnothing\}$ be a $\vee_2$-free family with $\mathrm{lu}_n(\mathcal{F}) \ge 2$. As $\vee_2$ is contained in C_3, we observe that no chain can intersect 3 elements of $\mathcal{F}$, so $\mathrm{lu}_n(\mathcal{F}) = 2$ and every full chain in B_n must intersect *exactly* 2 elements of $\mathcal{F}$. Let $\mathcal{C}$ be any full chain in B_n and suppose that $\mathcal{C} \cap \mathcal{F} = \{F_1, F_2\}$ with $F_1 \subset F_2$. Now choose $\mathcal{C}'$ to be a full chain that agrees with $\mathcal{C}$ through F_1 and avoids F_2 (note that $\mathcal{C}'$ can be found as $F_2 \ne [n]$). Therefore, $\mathcal{C}' \cap \mathcal{F} = \{F_1, F_3\}$ for some $F_3 \ne F_2$. As $\mathcal{C}$ and $\mathcal{C}'$ agree through F_1, it must be that $F_1 \subset F_3$, so $F_1F_2F_3$ forms a copy of $\vee_2$; a contradiction. Thus, $L_n'(\vee_2) < 2$ for every n.

Now let c be any t-coloring of B_{2t+1}, and, without loss of generality, suppose that $c(\varnothing) = 1$. Notice that if c restricted to $B_{2t+1} \setminus \{\varnothing\}$ admits a copy of $\vee_2$ in color 1, then c admits a copy of $\bowtie$ in color 1. For $i \in [t]$, define $\mathcal{F}_i = c^{-1}(i) \setminus \{[2t+1], \varnothing\}$; thus, by linearity,

$$2t = \mathrm{lu}_{2t+1}(B_{2t+1} \setminus \{[2t+1], \varnothing\}) = \mathrm{lu}_{2t+1}\left(\bigcup_{i=1}^{t} \mathcal{F}_i\right) = \sum_{i=1}^{t} \mathrm{lu}_{2t+1}(\mathcal{F}_i).$$

As noted earlier, $L'_n(\bowtie) = 2$ and $L'_n(\vee_2) < 2$ for any n, so $L'_{2t+1}(\vee_2) + \sum_{i=2}^{t} L'_{2t+1}(\bowtie) < 2t$. Therefore, there is either some $i \in \{2, \ldots, t\}$ such that $\mathcal{F}_i$ admits a copy of $\bowtie$ or $\mathcal{F}_1$ admits a copy of $\vee_2$. In any case, we arrive at a monochromatic copy of $\bowtie$. $\square$

Notice that the proof of the lower bound extends to show that for any posets $P_2, \ldots, P_t$,

$$\mathrm{BR}^1(\bowtie, P_2, \ldots, P_t) \geq 3 + \sum_{i=2}^{t} e(P_i),$$

even though $e(\bowtie) = 2$. Even though the lower bound given in Proposition 2.1 is not always tight, we conjecture that it is *essentially* correct in the following sense.

Conjecture 2.6. *Let P be a poset and let M be the least integer such that $P \subseteq B_M$. For any $t \geq 1$,*

$$\mathrm{BR}^1_t(P) = (t-1)e(P) + M.$$

Slightly weaker than this conjecture, we believe that if $P_1, \ldots, P_t$ are posets where P_i has height h_i and $P_i \subseteq B_{h_i - 1}$, then $\mathrm{BR}^1(P_1, \ldots, P_t) = \sum_{i=1}^{t}(h_i - 1)$. In order to confirm this belief, it suffices to show that $\mathrm{BR}^1(B_n, B_m) = n + m$. We verified this for small values of n and m; see Section 4.

To investigate the Boolean Ramsey numbers of posets with size much larger than their height, we consider a structure that is *wider* than a single chain and use that to consider an extension of Lubell functions. We use an idea that Grósz, Methuku and Tompkins [19] used to approach the Turán-type question. Let $A \subseteq B$ and define the *interval from A to B*, denoted $[A, B]$, to be the collection of sets C where $A \subseteq C \subseteq B$; we say the interval $[A, B]$ has *height m* if $m = |B \setminus A|$. For a full chain $\mathcal{A} = (A_0, \ldots, A_n)$ in B_n where $\varnothing = A_0 \subset A_1 \subset \cdots A_n = [n]$, define the *$m$-interval chain* $\mathcal{C}_m(\mathcal{A})$ as

$$\mathcal{C}_m(\mathcal{A}) = \bigcup_{i=0}^{n-m} [A_i, A_{i+m}].$$

Grósz, Methuku and Tompkins [19] proved $|\mathcal{C}_m(\mathcal{A})| = (n-m+2)2^{m-1}$ for all m-interval chains $\mathcal{C}_m(\mathcal{A})$.

For a family $\mathcal{F} \subseteq B_n$ and $1 \leq m \leq n$, define the *m-interval Lubell function of $\mathcal{F}$*, denoted $\mathrm{lu}_n^{(m)}(\mathcal{F})$, as

$$\mathrm{lu}_n^{(m)}(\mathcal{F}) = \frac{1}{n!} \sum_{\mathcal{A}} |\mathcal{F} \cap \mathcal{C}_m(\mathcal{A})|$$

where the sum is taken over all full chains $\mathcal{A}$. Observe that $\mathrm{lu}_n^{(1)}(\mathcal{F}) = \mathrm{lu}_n(\mathcal{F})$. For a poset P define $L_n^{(m)}(P) = \max\{\mathrm{lu}_n^{(m)}(\mathcal{F}) : \mathcal{F} \subseteq B_n,\ \mathcal{F} \text{ is } P\text{-free}\}$.

Due to the size of an m-interval chain, $\mathrm{lu}_n^{(m)}(B_n) = (n-m+2)2^{m-1}$. With this in mind, we arrive at a direct extension of Lemma 2.3.

Proposition 2.7. Let $P_1, \ldots, P_t$ be posets. If $\sum_{i=1}^{t} L_n^{(m)}(P_i) < (n-m+2)2^{m-1}$ for some m with $1 \leq m \leq n$, then $\mathrm{BR}^1(P_1, \ldots, P_t) \leq n$.

In order to apply Proposition 2.7 to attain a general upper bound on 1-uniform Boolean Ramsey numbers, we will make use of the following result.

Theorem 2.8 (Grósz, Methuku and Tompkins [19, Lemma 12])**.** *For $m \geq 2$, if P is a poset of height $h(P)$ and $\mathcal{F}$ is P-free, then for any m-interval chain $\mathcal{C}_m(\mathcal{A})$,*

$$|\mathcal{F} \cap \mathcal{C}_m(\mathcal{A})| \leq |P| - 1 + (h(P) - 1)(3m - 5)2^{m-2}.$$

We now provide a general upper bound on the 1-uniform Boolean Ramsey number for posets whose sizes are large compared to their heights.

Theorem 2.9. *Let $P_1, \ldots, P_t$ be posets, $S = \sum_{i=1}^{t}(|P_i| - 1)$, and $H = \sum_{i=1}^{t}(h(P_i) - 1)$.*

$$\mathrm{BR}^1(P_1, \ldots, P_t) \leq \left(\frac{3}{2}H + 1\right)\left(\lg\left(\frac{S}{H}\right) + 1\right).$$

Proof. If $n = \mathrm{BR}^1(P_1, \ldots, P_t) - 1$, then $\sum_{i=1}^{t} L_n^{(m)}(P_i) \geq (n - m + 2)2^{m-1}$ whenever $1 \leq m \leq n$ by Proposition 2.7. Therefore, by Theorem 2.8,

$$(n - m + 2)2^{m-1} \leq \sum_{i=1}^{t} L_n^{(m)}(P_i) \leq S + (3m - 5)2^{m-2}H$$

for all $2 \leq m \leq n$. As $S \geq H$, $\lg\left(\frac{S}{H}\right) \geq 0$, so set $m = \left\lfloor 2 + \lg\left(\frac{S}{H}\right)\right\rfloor$. Hence,

$$\begin{aligned} n &\leq \left(\frac{3}{2}H + 1\right)m - \frac{5}{2}H + 2^{1-m}S - 2 \\ &= \left(\frac{3}{2}H + 1\right)\left\lfloor 2 + \lg\left(\frac{S}{H}\right)\right\rfloor - \frac{5}{2}H + 2^{1-\left\lfloor 2+\lg\left(\frac{S}{H}\right)\right\rfloor}S - 2 \\ &\leq \left(\frac{3}{2}H + 1\right)\left(2 + \lg\left(\frac{S}{H}\right)\right) - \frac{5}{2}H + 2^{-\lg\left(\frac{S}{H}\right)}S - 2 \\ &= \frac{3}{2}H\lg\left(\frac{S}{H}\right) + \frac{3}{2}H + \lg\left(\frac{S}{H}\right) < \left(\frac{3}{2}H + 1\right)\left(\lg\left(\frac{S}{H}\right) + 1\right). \end{aligned}$$

□

A direct application of Theorem 2.9 presents reasonable bounds on the Boolean Ramsey number of various poset families

Corollary 2.10. *For positive integers $r, r_1, \ldots, r_t, s_1, \ldots, s_t$,*

$$\mathrm{BR}_t^1(B_r) \leq \left(\frac{3}{2}rt + 1\right)\left(\lg\left(\frac{2^r - 1}{r}\right) + 1\right) = O(r^2 t),$$

$$\mathrm{BR}^1(\bowtie_{s_1}^{r_1}, \ldots, \bowtie_{s_t}^{r_t}) \leq \left(\frac{3}{2}t + 1\right)\left(\lg\left(\frac{1}{t}\sum_{i=1}^{t}(r_i + s_i - 1)\right) + 1\right),$$

$$\mathrm{BR}^1(\vee_{r_1}, \ldots, \vee_{r_s}, \wedge_{r_{s+1}}, \ldots, \wedge_{r_t}) \leq \left(\frac{3}{2}t + 1\right)\left(\lg\left(\frac{1}{t}\sum_{i=1}^{t} r_i\right) + 1\right), \textit{ and}$$

$$\mathrm{BR}^1(\Diamond_{r_1}, \ldots, \Diamond_{r_t}) \leq (3t + 1)\lg\left(\frac{1}{t}\sum_{i=1}^{t}(r_i + 1)\right).$$

Corollary 2.10 brings to light an interesting phenomenon. If $P_1, \ldots, P_t$ are posets of bounded height ($h(P_i) \leq h_0$ for some constant h_0 not dependent on t) and $s = \frac{1}{t}\sum_{i=1}^{t}|P_i|$, then

$$\mathrm{BR}^1(P_1, \ldots, P_t) \leq O(t \log s).$$

Additionally, it is not hard to observe that $\diamondsuit_r$ is not contained in the union of k consecutive levels of the Boolean lattice whenever $r \geq 2^{k-1} - 1$. As such, $e(\diamondsuit_r) \geq \lfloor \lg(r+1) \rfloor + 1$, and we thus find that $\mathrm{BR}^1_t(\diamondsuit_r) = \Theta(t \log r)$.

Unfortunately, Theorem 2.9 does not allow us to prove that $\mathrm{BR}^1_t(B_r) = rt$ as implied by Conjecture 2.6; however, we know that the Ramsey number is at most quadratic in r, which is much better than the naïve bound of $(2^r - 1)t$.

3. 2-Uniform Boolean Ramsey Numbers

We now focus on 2-uniform partially-ordered Ramsey numbers. Due to recent interest in ordered Ramsey numbers, we will also include results concerning chain Ramsey numbers. We also will state our results in the k-uniform case when possible.

Proposition 3.1. Let $G_1, \ldots, G_t$ be k-uniform pographs.

$$\lg \mathrm{CR}^k(G_1, \ldots, G_t) \leq \mathrm{BR}^k(G_1, \ldots, G_t) \leq \mathrm{CR}^k(G_1, \ldots, G_t) - 1.$$

Proof. Let $N = \mathrm{CR}^k(G_1, \ldots, G_t)$. Observe that the chain C_N is contained in the Boolean lattice B_{N-1}, so any t-coloring of the k-chains in B_{N-1} contains a copy of G_i in the color i for some $i \in [t]$ and hence $\mathrm{BR}^k(G_1, \ldots, G_t) \leq N - 1$.

Let c be a t-coloring of the k-chains in C_{N-1} that avoids copies of G_i in color i for all $i \in [t]$. Let $n = \lfloor \lg N - 1 \rfloor$ and consider a linear extension $\pi : B_n \to C_{2^n}$. Thus, we can t-color every k-chain $A \in B_n$ using the color $c(\pi(A))$, where $\pi(A)$ is a k-chain in $C_{2^n} \subseteq C_{N-1}$. Since c avoids copies of G_i in color i for all $i \in [t]$, this coloring also avoids these copies in B_n. □

Let $G_1, \ldots, G_t$ be k-uniform pographs. If every linear extension of G_i is isomorphic for all $i \in [t]$, then observe that $\mathrm{CR}^k(G_1, \ldots, G_t) = \mathrm{OR}^k(G_1, \ldots, G_t)$; pographs with this property include $\vee_r$, $\wedge_r$, $\diamondsuit_r$, and $\bowtie^s_r$. When every G_i is totally-ordered, we have another equivalence of partially-ordered Ramsey numbers.

Proposition 3.2. If $G_1, \ldots, G_t$ are totally-ordered k-uniform pographs, then

$$\mathrm{BR}^k(G_1, \ldots, G_t) = \mathrm{CR}^k(G_1, \ldots, G_t) - 1 = \mathrm{OR}^k(G_1, \ldots, G_t) - 1.$$

Proof. The inequality $\mathrm{BR}^k(G_1, \ldots, G_t) \leq \mathrm{CR}^k(G_1, \ldots, G_t) - 1$ follows from Proposition 3.1.

Let $N = \mathrm{CR}^k(G_1, \ldots, G_t) - 1$ and let c be a t-coloring of the k-chains in C_N that does not contain a copy of G_i in color i for all $i \in [t]$. Define a map $\rho : B_{N-1} \to C_N$ by $\rho(A) = |A| + 1$; if $A_1 \subset A_2 \subset \cdots A_\ell$ is an ℓ-chain in B_{N-1}, then $(\rho(A_1), \ldots, \rho(A_\ell))$ is an ℓ-chain in C_N. For a k-chain $A_1 \subset A_2 \subset \cdots \subset A_k$ in the Boolean lattice B_{N-1}, let $c'(A_1, \ldots, A_k) = c(\rho(A_1), \rho(A_2), \ldots, \rho(A_k))$. Consider a copy of G_i in B_{N-1}. Since G_i is totally-ordered, the elements of G_i form a chain in B_{N-1} and thus ρ maps the elements of G_i onto a copy of G_i in C_N. Since c avoids i-colored copies of G_i in C_N, so does c' avoid i-colored copies of G_i in B_{N-1}. □

The above argument requires that the vertices of a totally-ordered graph occupy distinct levels in any embedding of G into the Boolean lattice. If G is not totally-ordered, then there is a pair of vertices which are incomparable; these two vertices may occupy the same level in an embedding of G into B_n. It seems reasonable to expect that if G contains large antichains, then the lower bound in Proposition 3.1 should be closer to the truth. We find this to be true for a few classes of pographs with large antichains.

3.1. Matchings.

A natural class of k-uniform pographs with large antichains are those were the k-chains are completely independent. In the case of k-uniform matchings, we find the logarithmic bound on the Boolean Ramsey number is essentially tight.

Theorem 3.3. *Let $m_1 \geq \cdots \geq m_t$ and let $M^k_{m_1}, \ldots, M^k_{m_t}$ be k-uniform matchings of size $m_1, \ldots, m_t$.*

$$\lg\left(km_1 + \sum_{i=2}^{t}(m_i - 1)\right) \leq \mathrm{BR}^k(M^k_{m_1}, \ldots, M^k_{m_t}) \leq \left\lceil \lg\left(1 + \sum_{i=1}^{t}(m_i - 1)\right)\right\rceil + k - 1.$$

Proof. Lower bound. Observe $\mathrm{CR}^k(M^k_{m_1}, \ldots, M^k_{m_t}) = \mathrm{R}^k(M^k_{m_1}, \ldots, M^k_{m_t})$ as every copy of an unordered matching can be considered a linear extension of a partially-ordered matching. Alon, Frankl, and Lovász [1] demonstrated that if $m_1 \geq \cdots \geq m_t$, then $\mathrm{R}^k(M^k_{m_1}, \ldots, M^k_{m_t}) = km_1 + \sum_{i=1}^{t}(m_i - 1)$. Apply Proposition 3.1 to complete the lower bound.

Upper bound. Let $N = \left\lceil \lg\left(1 + \sum_{i=1}^{t}(m_i - 1)\right)\right\rceil$ and let c be a t-coloring of the k-chains in B_{N+k-1}. Let X be the family of subsets of $[N]$ within B_{N+k-1}. For every set $A \in X$, define the *extension* of A to be the k-chain $\mathrm{ext}(A) = (A, A \cup \{N+1\}, A \cup \{N+2\}, \ldots, A \cup \{N+1, \ldots, N+k-1\})$. For $i \in [t]$, define the set T_i to be the sets $A \in X$ where $c(\mathrm{ext}(A)) = i$. Since $|X| = 2^N \geq 1 + \sum_{i=1}^{k}(m_i - 1)$, the pigeonhole principle implies that $|T_i| \geq m_i$ for some i. The collection of k-chains $\mathrm{ext}(A)$ for $A \in T_i$ form an i-colored matching of size at least m_i. □

Matchings are usually much simpler than other graphs. Indeed, we limit our focus to 2-uniform pographs for the remainder of this section.

3.2. Cups and Caps.

We now focus on the Boolean Ramsey numbers of r-caps and r-cups. To begin, the following proposition follows directly from the pigeonhole principle by considering all r_i-cups with minimum element $\varnothing$ or all r_i-caps with maximum element $[N]$.

Proposition 3.4. For positive integers $r_1, \ldots, r_t$,

$$\mathrm{BR}^2(\vee_{r_1}, \ldots, \vee_{r_t}) = \mathrm{BR}^2(\wedge_{r_1}, \ldots, \wedge_{r_t}) = \left\lceil \lg\left(2 + \sum_{i=1}^{t}(r_i - 1)\right)\right\rceil.$$

While the Boolean Ramsey number was simple to compute when considering a collection of cups or a collection of caps, the Ramsey numbers become more complicated when considering a collection of both cups and caps. This next proposition states that knowing the 2-color partially-ordered Ramsey number for cup verses cap is sufficient to determine the multicolor Ramsey number.

Proposition 3.5. Let $R = 1 + \sum_{i=1}^{n}(r_i - 1)$ and $S = 1 + \sum_{i=1}^{m}(s_i - 1)$.

$$\mathrm{CR}^2(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m}) = \mathrm{CR}^2(\vee_R, \wedge_S), \quad \text{and}$$
$$\mathrm{BR}^2(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m}) = \mathrm{BR}^2(\vee_R, \wedge_S).$$

Proof. We prove equality by demonstrating both inequalities.

($\leq$) Consider an $(n+m)$-coloring c of the edges of either a chain C_N or a Boolean lattice B_N. Let c' be a 2-coloring where $c'(e) = 1$ if $c(e) \leq n$ and $c'(e) = 2$ if $c(e) > n$. If c avoids i-colored copies of $\vee_{r_i}$ and $(n+j)$-colored copies of $\wedge_{s_j}$, then c' avoids 1-colored copies of $\vee_R$ and 2-colored copies of $\wedge_S$.

($\geq$) Let c be a 2-coloring of the edges of either a chain C_N or a Boolean lattice B_N and suppose that c does not contain a copy of $\vee_R$ in color 1 or a copy of $\wedge_S$ in color 2. We will construct an $(n+m)$-coloring c. Since c does not contain a 1-colored copy of $\vee_R$, we have $|\mathcal{U}_1(v)| < R$; partition $\mathcal{U}_1(v)$ into n parts $P_1 \cup \cdots \cup P_n$ such that $|P_i| \leq r_i - 1$ and let $c'(vu) = i$ if $u \in P_i$. Since c does not contain a 2-colored copy of $\wedge_S$, we have $|\mathcal{D}_2(v)| < S$; partition $\mathcal{D}_2(v)$ into m parts $P_1 \cup \cdots \cup P_m$ such that $|P_j| \leq s_j - 1$ and let $c'(vu) = n + j$ if $u \in P_j$. Every edge is colored exactly once by the process above and hence c' avoids i-colored copies of $\vee_{r_i}$ and $(n + j)$-colored copies of $\wedge_{s_j}$. □

Choudum and Ponnusamy [4] determined $\mathrm{CR}^2(\vee_r, \wedge_s)$ exactly.

Theorem 3.6 (Choudum and Ponnusamy [4])**.** *For integers $r, s \geq 2$,*

$$\mathrm{CR}^2(\vee_r, \wedge_s) = \left\lfloor \frac{\sqrt{1 + 8(r-1)(s-1)} - 1}{2} \right\rfloor + r + s.$$

Observe that this implies $\mathrm{CR}^2(\vee_r, \wedge_s) \leq (1 + \sqrt{2})(r + s)$. Therefore, by applying Proposition 3.5, we see that

$$\mathrm{CR}^2(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m}) = \left\lfloor \frac{\sqrt{1 + 8(R-1)(S-1)} - 1}{2} \right\rfloor + R + S \leq (1 + \sqrt{2})(R + S),$$

where $R = 1 + \sum_{i=1}^{n}(r_i - 1)$ and $S = 1 + \sum_{i=1}^{m}(s_i - 1)$.

In contrast to the linear bound of chain Ramsey numbers, the following theorem shows that the Boolean Ramsey numbers for cups and caps is logarithmic.

Theorem 3.7. *For integers $r, s \geq 2$,*

$$\lg\left(\left\lfloor \frac{\sqrt{1 + 8(r-1)(s-1)} - 1}{2} \right\rfloor + r + s\right) \leq \mathrm{BR}^2(\vee_r, \wedge_s) \leq \left\lceil \log_{3/2}(r + s - 1) \right\rceil.$$

Proof. Lower Bound. The lower bound follows from Theorem 3.6 and applying Proposition 3.1.

Upper Bound. Let $N = \left\lceil \log_{3/2}(r + s - 1) \right\rceil$ and suppose that c is a 2-coloring of the edges of B_N that avoids copies of $\vee_r$ in color 1 and avoids copies of $\wedge_s$ in color 2. Thus, for any $v \in B_N$, $|\mathcal{U}_1(v)| \leq r - 1$ and $|\mathcal{D}_2(v)| \leq s - 1$. In particular, this implies that $|\mathcal{D}_1(v)| = |\mathcal{D}(v)| - 1 - |\mathcal{D}_2(v)| \geq 2^{|v|} - s$.

Let $W = B_N \setminus \{[N]\}$ and let T be the set of elements v in W where $|\mathcal{U}_1(v) \cap W| = r - 1$. As c avoids copies of $\vee_r$ in color 1, for any $v \in T$, $c(v, [N]) = 2$. Hence, $|T| \leq s - 1$ since c avoids copies of $\wedge_s$ in color 2.

Let b be the number of edges uv with $c(uv) = 1$ and both u and v are in W, then

$$b = \sum_{v \in W} |\mathcal{D}_1(v)| \geq \sum_{v \in W} (2^{|v|} - s) = \sum_{i=0}^{n-1} \binom{N}{i} 2^i - s(2^N - 1) = 3^N - 2^N(s + 1) + s.$$

On the other hand,

$$\begin{aligned} b &= \sum_{v \in W} |\mathcal{U}_1(v) \cap W| \\ &= \sum_{v \in T}(r-1) + \sum_{v \in W \setminus T} |\mathcal{U}_1(v) \cap W| \\ &\leq |T|(r-1) + (2^N - 1 - |T|)(r-2) \\ &= |T| + (2^N - 1)(r-2) \\ &\leq s - 1 + (2^N - 1)(r-2). \end{aligned}$$

Therefore, $3^N - 2^N(s+1) + s \leq b \leq s - 1 + (2^N - 1)(r-2)$, so

$$\left(\frac{3}{2}\right)^N \leq r + s - 1 - (r-1)2^{-N} < r + s - 1.$$

This, however, is a contradiction as $N = \left\lceil \log_{3/2}(r+s-1) \right\rceil$. □

By applying Proposition 3.5, observe that

$$\lg\left(\left\lfloor \frac{\sqrt{1+8(R-1)(S-1)}-1}{2} \right\rfloor + R + S\right) \leq \mathrm{BR}^2(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m}) \leq \left\lceil \log_{3/2}(R+S-1) \right\rceil,$$

where $R = 1 + \sum_{i=1}^{n}(r_i - 1)$ and $S = 1 + \sum_{i=1}^{m}(s_i - 1)$.

3.3. **Diamonds.** An r-diamond combines the behavior of an r-cup with an r-cap. Despite doubling the number of edges in the pograph, we find similar logarithmic behavior in the Boolean Ramsey numbers. However, our methods focus on the 2-color case and fail to extend to the generic t-color case.

Using Theorem 3.6, Balko, Cibulka, Král and Kynčl [2] argued that $11 \leq \mathrm{CR}^2_2(\diamondsuit_2) \leq 13$ and show that the lower bound is tight with computer assistance. We apply their technique that yields an upper bound of 13 to attain an general upper bound for the chain Ramsey number of $\diamondsuit_r$.

Theorem 3.8. *If $r, s \geq 2$, then*

$$\mathrm{CR}^2(\diamondsuit_r, \diamondsuit_s) \leq 2 \cdot \left\lfloor \frac{\sqrt{1+8(r-1)(s-1)}-1}{2} \right\rfloor + 3(r+s) - 1$$

Proof. Let $N = 2 \cdot \left\lfloor \frac{\sqrt{1+8(r-1)(s-1)}-1}{2} \right\rfloor + 3(r+s) - 1$ and suppose that c is a 2-coloring of the edges of C_N that avoids copies of $\diamondsuit_r$ in color 1 and avoids copies of $\diamondsuit_s$ in color 2. Therefore, $|\mathcal{U}_1(1) \cap \mathcal{D}_1(N)| \leq r - 1$ and $|\mathcal{U}_2(1) \cap \mathcal{D}_2(N)| \leq s - 1$. Hence, $|\mathcal{U}_1(1) \cap \mathcal{D}_2(N)| + |\mathcal{U}_2(1) \cap \mathcal{D}_1(N)| \geq (N-2) - (r-1) - (s-1) = N - r - s$.

By the pigeonhole principle, there is some $i \in \{1,2\}$ for which

$$\begin{aligned}|\mathcal{U}_i(1) \cap \mathcal{D}_{3-i}(N)| &\geq \left\lceil \frac{N-r-s}{2} \right\rceil \\ &= \left\lceil \left\lfloor \frac{\sqrt{1+8(r-1)(s-1)}-1}{2} \right\rfloor + r + s - \frac{1}{2} \right\rceil \\ &= \left\lfloor \frac{\sqrt{1+8(r-1)(s-1)}-1}{2} \right\rfloor + r + s \\ &= \mathrm{CR}^2(\wedge_r, \vee_s).\end{aligned}$$

If this is true for $i=1$, then c restricted to $\mathcal{U}_1(1) \cap \mathcal{D}_2(N)$ must admit either a $\vee_s$ in color 2, in which case c admits a $\diamondsuit_s$ in color 2, or a $\wedge_r$ in color 1, in which case c admits a $\diamondsuit_r$ in color 1. A similar contradiction is found if the inequality holds for $i=2$. □

Corollary 3.9. *If $s, r \geq 2$, then* $\mathrm{OR}^2(\diamondsuit_s, \diamondsuit_r) = \mathrm{CR}^2(\diamondsuit_s, \diamondsuit_r) \leq (3+\sqrt{2})(r+s) \approx 4.414(r+s)$.

This upper bound is asymptotically correct, up to the leading constant.

Proposition 3.10. If $s \geq r \geq 2$, then $\mathrm{OR}^2(\diamondsuit_s, \diamondsuit_r) = \mathrm{CR}^2(\diamondsuit_s, \diamondsuit_r) > 2s+2$.

Proof. Let $N = 2s+2$ and consider $X_1 = \{1, \ldots, s+1\}$ and $X_2 = \{s+2, \ldots, N\}$. If an edge has both endpoints in X_i for some i, then color that edge with color 1. If an edge has one endpoint in X_1 and another in X_2, then color that edge with color 2. Observe that there is no $\diamondsuit_r$ in color 2, as there is no chain of length 2 in color 2. Further, there is no $\diamondsuit_s$ in color 1, as such a subgraph would be entirely contained in X_1 or X_2, but these sets have size $s+1$ and $|V(\diamondsuit_s)| = s+2$. □

Note that Proposition 3.10 immediately implies that if $s \geq r \geq 2$, then $\mathrm{BR}^2(\diamondsuit_s, \diamondsuit_r) \geq \lg(2s+3)$.

To investigate and upper bound on the Boolean Ramsey numbers of diamonds, we first consider diamonds and cups (Theorem 3.11) before completing the argument for two diamonds (Theorem 3.12).

Theorem 3.11. *Let $s, r \geq 2$ be integers.*

$$\mathrm{BR}^2(\diamondsuit_s, \vee_r) \leq \mathrm{BR}^2(\wedge_{s+r}, \vee_r) \leq \left\lceil \log_{3/2}(2r+s-1) \right\rceil$$

Proof. The second inequality holds by Theorem 3.7. Let $N = \mathrm{BR}^2(\wedge_{s+r}, \vee_r)$ and consider a 2-coloring of the edges of B_N and suppose the 2-coloring does not contain an s-diamond in color 1 or an r-cup in color 2. Therefore, there is an $(s+r)$-cap in color 1. Let $A_0, A_1, \ldots, A_{s+r-1}, B$ be the sets in this cap where $A_i \subseteq B$ for all i. If the empty set is in the cap, then let $A_0 = \varnothing$. There are $s+r$ edges from the empty set to the sets A_i with $i \in \{1, \ldots, s+r-1\}$. Since the coloring avoids r-cups in color 2, there must be at least s sets A_i such that the edge $(\varnothing, A_i)$ has color 1. Thus, these A_i's along with the empty set and B forms an s-diamond of color 1; a contradiction. □

Theorem 3.12. *Let $s, r \geq 2$ be integers.*

$$\mathrm{BR}^2(\diamondsuit_s, \diamondsuit_r) \leq \mathrm{BR}^2(\diamondsuit_s, \vee_{s+r-1}) + \lceil \lg(2s+2r) \rceil \leq 2\left\lceil \log_{3/2}(2r+2s-1) \right\rceil.$$

Proof. The second inequality holds by Theorem 3.11 and logarithmic identities. Let $N = \mathrm{BR}^2(\diamondsuit_s, \vee_{s+r-1})$ and $M = \lceil \lg(2s+2r) \rceil$. Consider a 2-coloring c of the edges of B_{N+M}. Suppose for the sake of contradiction that c does not contain an s-diamond in color 1 and does not contain an r-diamond in color 2.

For $j \in \{1,2\}$, let I_j contain the sets Z such that $[N] \subset Z \subset [N+M]$. and $c(Z,[N+M]) = j$. Since $|I_1 \cup I_2| = 2^M - 2 \geq 2s+2r-2$, either $|I_1| \geq s+r-1$ or $|I_2| \geq s+r-1$. We will assume that $|I_1| \geq s+r-1$; the other case follows by a symmetric argument. Let $I \subseteq I_1$ with $|I| = s+r-1$.

Since $N = \mathrm{BR}^2(\Diamond_s, \vee_{s+r-1})$, and c does not contain an s-diamond in color 1, there exists an $(s+r-1)$-cup of color 2 in $\mathcal{D}([N])$. Let $A_0, B_1, \ldots, B_{s+r-1}$ be the sets of this cup such that $A_0 \subset B_j$ for each $j \in \{1,\ldots,s+r-1\}$. Notice that $B_j \subset Z$ for all $j \in \{1,\ldots,s+r-1\}$ and all $Z \in I$. The edges between $B_1,\ldots,B_{s+r-1}$ and the sets $Z \in I$ form a 2-colored copy of the complete bipartite graph $K_{s+r-1,s+r-1}$.

For every $j \in \{1,\ldots,s+r-1\}$ there are at most $s-1$ edges of color 1 from B_j to the sets $Z \in I$, since c avoids s-diamonds in the color 1. For every $Z \in I$, there are at most $r-1$ edges of color 2 from the sets $B_1,\ldots,B_{s+r-1}$ to Z, since c avoids r-diamonds in the color 2. However, this implies that the total number of edges in this complete bipartite graph is at most $(s+r-1)((s-1)+(r-1)) < (s+r-1)^2$, a contradiction. □

Using Theorems 3.11 and 3.12, we find $\mathrm{BR}^2(\Diamond_2, \vee_3) \leq 5$ and $\mathrm{BR}^2(\Diamond_2, \Diamond_2) \leq 8$. With a more specialized argument for the case $r = s = 2$, one can prove $\mathrm{BR}^2(\Diamond_2, \Diamond_2) \leq \mathrm{BR}^2(\Diamond_2, \vee_3) + 2$, but this is not tight. In the next section, we discuss computational methods to compute Boolean Ramsey numbers, and we verify that $\mathrm{BR}^2(\Diamond_2, \vee_3) = 4$ and $\mathrm{BR}^2(\Diamond_2, \Diamond_2) = 5$.

4. Computational Results

Ramsey numbers are difficult to compute in all but the simplest of cases. A naïve algorithm for testing $\mathrm{R}_t^k(G) > n$ takes $O(t^{n^k})$ steps, and advanced algorithm techniques do not improve on the asymptotic growth of this method. However, using the same method to test $\mathrm{BR}_t^k(G) > n$ can require $O(t^{(k+1)^n})$ steps. In fact, simply storing a t-coloring of the k-chains in B_n requires $(k+1)^n \lg t$ bits of space. This makes finding exact values of 2-color, 2-uniform Boolean Ramsey numbers very difficult once $n \geq 5$.

To test if $\mathrm{BR}^2(H_1, H_2) > n$, we use a SAT formulation to determine if there exists a 2-coloring c of the comparable pairs in B_n that avoids copies of H_1 in color 1 and avoids copies of H_2 in color 2. For every comparable pair $A \subset B$, we let $x_{A,B}$ be a Boolean variable; the variable $x_{A,B}$ is true exactly when $c(A,B) = 1$. For every copy of H_1 in B_n, we create a constraint that requires at least one variable $x_{A,B}$ to be false among the edges (A,B) in the copy of H_1. Similarly, for every copy of H_2 in B_n, we create a constraint that requires at least one variable $x_{A,B}$ to be true among the edges (A,B) in the copy of H_2. There exists such a 2-coloring if and only if these constraints can be simultaneously satisfied.

We used a similar SAT formulation to demonstrate that $\mathrm{BR}^1(B_n, B_m) > n+m-1$ (formulation is satisfiable) and $\mathrm{BR}^1(B_n, B_m) \leq n+m$ (formulation is unsatisfiable) when $3 \geq n \geq m \geq 1$.

We used Sage [28] to construct our SAT formulations in SMT2 format. We then used the Microsoft Z3 [11] SMT[2] solver to test the formulations. The results are summarized in Table 1. These computations were completed using a standard laptop computer with each test taking at most a few hours. All Sage code and SAT formulations are available online[3].

This method was limited by the exponential growth in the size of the formulations more than the time it takes to solve them. We selected only a few examples to test with $n = 5$ due to the number of copies of the

[2]Satisfiability Modulo Theory.
[3]See `http://orion.math.iastate.edu/dstolee/data.htm` for all code and data.

(a) $\mathrm{BR}^2(\vee_r, \wedge_s)$.

	$\wedge_2$	$\wedge_3$	$\wedge_4$	$\wedge_5$
$\vee_2$	3	3	4	4
$\vee_3$	3	4	5	5
$\vee_4$	4	5	5	
$\vee_5$	4	5		

(b) $\mathrm{BR}^2(H, \wedge_s)$.

	$\wedge_2$	$\wedge_3$	$\wedge_4$
$\diamondsuit_2$	4	4	5
B_2	4	4	5
$\diamondsuit_3$	4	5	5

(c) $\mathrm{BR}^2(H_1, H_2)$.

	$\diamondsuit_2$	B_2
$\diamondsuit_2$	5	6
B_2	6	6

(d) $\mathrm{BR}^2(\vee_r, M_s)$.

	M_2	M_3	M_4
$\vee_2$	3	3	4
$\vee_3$	3	4	4
$\vee_4$	4	4	4
$\vee_5$	4	4	4
$\vee_6$	4	4	4

(e) $\mathrm{BR}^2(M_r, M_s)$.

	M_2	M_3	M_4
M_2	3	3	4
M_3		3	4
M_4			4

(f) $\mathrm{BR}^2(C_r, M_s)$

	M_2	M_3	M_4
C_2	2	3	3
C_3	3	4	4
C_4	4	5	
C_5	5		

(g) $\mathrm{BR}^2(C_r, \wedge_s) = \mathrm{BR}^2(C_r, \vee_s)$

	$\wedge_2$	$\wedge_3$	$\wedge_4$
C_2	2	2	3
C_3	3	3	4
C_4	4	4	5

(h) $\mathrm{BR}^2(\vee_r, \bowtie_m^n) = \mathrm{BR}^2(\wedge_r, \bowtie_n^m)$

	$\bowtie_2^2$	$\bowtie_2^3$	$\bowtie_3^2$	$\bowtie_3^3$
$\vee_2$	4	4	4	5
$\vee_3$	4	4	5	5
$\vee_4$	4	5		

(i) $\mathrm{BR}^2(\vee_r, W_s) = \mathrm{BR}^2(\wedge_r, W_s)$

	W_2	W_3	W_4	W_5
$\vee_2$	4	4	4	4
$\vee_3$	4	4	4	4
$\vee_4$	4	5		

Table 1. Computational results for small 2-uniform Boolean Ramsey numbers.

pographs H_1 and H_2 that appeared within B_5. We could test $\mathrm{BR}^2(B_2, \diamondsuit_2) = \mathrm{BR}^2(B_2, B_2) = 6$ due to the fact that B_2 and $\diamondsuit_2$ have only four elements, which greatly limited the number of copies appear within B_6, but these tests were our largest computations.

A highly specialized algorithm may be able to extend these results to more examples when $n = 6$, but we expect this will be very difficult.

5. Other Poset Families

While we have mainly focused on chain Ramsey numbers and Boolean Ramsey numbers, many other families of posets can give rise to interesting Ramsey numbers.

5.1. **Generic Poset Families.** Let $\mathcal{P} = \{P_n : n \geq 1\}$ be a poset family with $P_n \subseteq P_{n+1}$ for all n. For a t-tuple $(G_1, \dots, G_t)$ of k-uniform pographs, we say that $\mathcal{P}$ is *k-Ramsey for* $(G_1, \dots, G_t)$ if there exists an N such that every t-coloring of the k-chains in P_N contains an i-colored copy of G_i for some i. The partially-ordered Ramsey number $\mathrm{R}^k_{\mathcal{P}}(G_1, \dots, G_t)$ exists exactly when $\mathcal{P}$ is k-Ramsey for $(G_1, \dots, G_t)$.

We say a family $\mathcal{P}$ is a *universal poset family* if $\mathcal{P}$ is k-Ramsey for every t-tuple of k-uniform pographs and every $k \geq 1$. If the height of P_n grows without bound, then $\mathcal{P}$ is a universal poset family as eventually P_n contains a chain of order $\mathrm{CR}^k(G_1, \dots, G_t)$ for any $G_1, \dots, G_t$. Some of our results hold for universal poset families, such as Propositions 3.5. Other results must be generalized slightly, such as the following generalization of Proposition 3.1.

Proposition 5.1. Let $\mathcal{P} = \{P_n : n \geq 1\}$ be a universal poset family. Define $s_{\mathcal{P}}(n)$ to be the minimum N such that $|P_N| \geq n$. Define $h_{\mathcal{P}}(n)$ to be the minimum N such that $C_n \subseteq P_N$. Then,

$$s_{\mathcal{P}}(\mathrm{CR}^k(G_1, \dots, G_t)) \leq \mathrm{R}^k_{\mathcal{P}}(G_1, \dots, G_t) \leq h_{\mathcal{P}}(\mathrm{CR}^k(G_1, \dots, G_t)).$$

Using the function $h_{\mathcal{P}}(n)$, one can restate Proposition 3.2 as $\mathrm{R}^k_{\mathcal{P}}(G_1,\dots,G_t) = h_{\mathcal{P}}(\mathrm{CR}^k(G_1,\dots,G_t))$ for totally-ordered graphs $G_1,\dots,G_t$.

5.2. **Rooted Bipartite Ramsey Numbers.** A poset family does not need to be universal in order to be interesting. Consider the family $\mathcal{K} = \{\bowtie^n_n : n \geq 1\}$ of n,n-butterfly posets. This family is not universal since $C_3 \not\subseteq \bowtie^n_n$ for any n. However, we can still consider $G_1,\dots,G_t$ to be pographs whose posets partition into two antichains $V(G_i) = X_i \cup Y_i$ where every $x \in X_i$ is comparable to at least one element $y \in Y_i$ with $x \leq y$. In this case, the Ramsey number $\mathrm{R}^2_{\mathcal{K}}(G_1,\dots,G_t)$ is the minimum N such that every t-coloring of the edges of the complete bipartite graph $K_{N,N}$ with vertex set $V(K_{N,N}) = A \cup B$ contains an i-colored copy of the bipartite graph G_i where $X_i \subseteq A$ and $Y_i \subseteq B$ for some i.

If we remove the condition that $X_i \subseteq A$ and $Y_i \subseteq B$, then this Ramsey problem is identical to finding *bipartite Ramsey numbers* (see [3, 6, 15, 21, 22]). The equivalent of the Turán problem in this context is called the *Zarenkiewicz problem* (see [14, 15, 22]). The most widely studied version of these numbers are those where $G_i = \bowtie^m_n$ for some n, m.

With the condition that $X_i \subseteq A$ and $Y_i \subseteq B$, we can call $\mathrm{R}^2_{\mathcal{K}}(G_1,\dots,G_t)$ the *rooted* bipartite Ramsey number. In this case, it may be true that $\mathrm{R}^2_{\mathcal{K}}(\bowtie^s_r, \bowtie^s_r) \neq \mathrm{R}^2_{\mathcal{K}}(\bowtie^s_r, \bowtie^r_s)$ when $r \neq s$. The final paragraph of the proof of Thereom 3.12 implicitly proves and uses the fact that $\mathrm{R}^2_{\mathcal{K}}(\wedge_s, \vee_r) = \mathrm{R}^2_{\mathcal{K}}(\vee_s, \vee_r) = s + r - 1$.

5.3. **High-Dimensional Grids.** Closely related to the Boolean lattice is the *m-dimensional ℓ-grid* $([\ell]^m, \preceq)$, whose elements are m-tuples $(x_1,\dots,x_n)$ where every coordinate x_i is in the set $[\ell]$, and $(x_1,\dots,x_n) \preceq (y_1,\dots,y_n)$ if and only if $x_i \leq y_i$ for all i (in particular, the Boolean lattice B_n corresponds to $[2]^n$). When constructing a universal poset family $\mathcal{P} = \{P_n : n \geq 1\}$ from these grids, we have two natural options for the parameter n. First, we could have the dimension grow with n; let $Q_n(\ell) = [\ell]^n$ and $\mathcal{Q}(\ell) = \{Q_n(\ell) : n \geq 1\}$. Second, we could have the length grow with n; let $H_n(m) = [n]^m$ and $\mathcal{H}(m) = \{H_n(m) : n \geq 1\}$. Along these lines, we provide analogues of theorems from Section 3 for each of these cases.

Theorem 5.2 (Analogue of Theorem 3.7)**.** *For $s, r \geq 2$,*

$$\log_\ell\left(\left\lfloor \frac{\sqrt{1+8(r-1)(s-1)}-1}{2}\right\rfloor + r + s\right) \leq \mathrm{R}^2_{\mathcal{Q}(\ell)}(\vee_r, \wedge_s) \leq \left\lceil \log_{(\ell+1)/2}(r+s-1)\right\rceil \quad \text{and}$$

$$\left(\left\lfloor \frac{\sqrt{1+8(r-1)(s-1)}-1}{2}\right\rfloor + r + s\right)^{1/n} \leq \mathrm{R}^2_{\mathcal{H}(m)}(\vee_r, \wedge_s) \leq \left\lceil 2(r+s-1)^{1/n}\right\rceil - 1.$$

Theorem 5.3 (Analogue of Theorem 3.11)**.** *For $s, r \geq 2$,*

$$\mathrm{R}^2_{\mathcal{Q}(\ell)}(\Diamond_s, \vee_r) \leq \mathrm{R}^2_{\mathcal{Q}(\ell)}(\wedge_{s+r}, \vee_r) \leq \left\lceil \log_{(\ell+1)/2}(2r+s-1)\right\rceil \quad \text{and}$$

$$\mathrm{R}^2_{\mathcal{H}(m)}(\Diamond_s, \vee_r) \leq \mathrm{R}^2_{\mathcal{H}(m)}(\wedge_{s+r}, \vee_r) \leq \left\lceil 2(2r+s-1)^{1/n}\right\rceil - 1.$$

Theorem 5.4 (Analogue of Theorem 3.12)**.** *For $s, r \geq 2$,*

$$\mathrm{R}^2_{\mathcal{Q}(\ell)}(\Diamond_s, \Diamond_r) \leq \mathrm{R}^2_{\mathcal{Q}(\ell)}(\Diamond_r, \vee_{s+r-1}) + \lceil \log_\ell(2s+2r)\rceil \leq 2\left\lceil \log_{(\ell+1)/2}(2r+2s-1)\right\rceil \quad \text{and}$$

$$\mathrm{R}^2_{\mathcal{H}(m)}(\Diamond_s, \Diamond_r) \leq \mathrm{R}^2_{\mathcal{H}(m)}(\Diamond_s, \vee_{s+r-1}) + \left\lceil (2s+2r)^{1/n}\right\rceil \leq 3\left\lceil (2r+2s-1)^{1/n}\right\rceil.$$

The proof of each of these theorems are identical to their analogues in the Boolean lattice. Notice that in each case, the Ramsey number is within a constant factor of the lower bound given in Proposition 5.1. It would be of interest to explore other partially-ordered Ramsey numbers using $\mathcal{Q}(\ell)$ or $\mathcal{H}(m)$ as the host family.

6. Future Work

For 1-uniform Boolean Ramsey numbers, the main open question is Conjecture 2.6. It is important to point out that Theorem 2.9 employed only the bound $\mathrm{lu}_n^{(m)}(\mathcal{F}) \leq \max_{\mathcal{C}_m} |\mathcal{F} \cap \mathcal{C}_m|$. It would be interesting to explore the actual value of $L_n^{(m)}(P)$ for specific posets P.

We are particularly interested in the properties of partially-ordered graphs whose Boolean Ramsey numbers are within a constant factor of the lower bound given in Proposition 3.1. In particular, we ask the following.

Question 6.1. *What properties must a graph G have so that the lower bound on the Boolean Ramsey number of G given in Proposition 3.1 is tight up to a constant?*

We suspect that the answer to this question will focus on the properties of the underlying poset of G and have very little to do with the actual edges of G. In particular, we suspect that the answer relies heavily on the number and/or size of the antichains in the underlying poset.

Previously, as far as we are aware, other authors have, for the most part, only been interested in the Lubell function of a P-free family when approaching the Turán problem in the Boolean lattice. To this end, many Lubell functions have been ignored if they do not provide the desired bound in the Turán problem. We think that an exploration of the Lubell functions of P-free families is interesting in and of itself. In particular, we are interested in attaining good upper bounds on $L_n(B_d)$. Beyond this, further exploration of the m-interval Lubell function of P-free families may provide interesting insights into both the Ramsey and Turán problems in the Boolean lattice.

Returning to 2-uniform Boolean Ramsey numbers, an exploration of $\mathrm{BR}_t^2(B_n)$ would be of great interest. By applying the bounds on $\mathrm{R}_2^2(K_n)$, we immediately observe that $\Omega(2^{n/2}) \leq \mathrm{BR}_2^2(B_n) \leq O(4^{2^n})$. We believe the upper bound to be far from the truth and would expect only an exponential bound, but any improvement to either bound would be of interest.

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