Rainbow arithmetic progressions

Steve Butler[1] Craig Erickson[1] Leslie Hogben[1,2] Kirsten Hogenson[1]
Lucas Kramer[1] Richard L. Kramer[1] Jephian Chin-Hung Lin[1]
Ryan R. Martin[1,3] Derrick Stolee[1,4] Nathan Warnberg[1] Michael Young[1,5,6]

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Abstract

In this paper, we investigate the anti-Ramsey (more precisely, anti-van der Waerden) properties of arithmetic progressions. For positive integers n and k, the expression $\mathrm{aw}([n], k)$ denotes the smallest number of colors with which the integers $\{1, \ldots, n\}$ can be colored and still guarantee there is a rainbow arithmetic progression of length k. We establish that $\mathrm{aw}([n], 3) = \Theta(\log n)$ and $\mathrm{aw}([n], k) = n^{1-o(1)}$ for $k \geq 4$.

For positive integers n and k, the expression $\mathrm{aw}(\mathbb{Z}_n, k)$ denotes the smallest number of colors with which elements of the cyclic group of order n can be colored and still guarantee there is a rainbow arithmetic progression of length k. In this setting, arithmetic progressions can "wrap around," and $\mathrm{aw}(\mathbb{Z}_n, 3)$ behaves quite differently from $\mathrm{aw}([n], 3)$, depending on the divisibility of n. In fact, $\mathrm{aw}(\mathbb{Z}_{2^m}, 3) = 3$ for any positive integer m. However, for $k \geq 4$, the behavior is similar to the previous case, that is, $\mathrm{aw}(\mathbb{Z}_n, k) = n^{1-o(1)}$.

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1 Introduction

Let G be an additive (abelian) group such as the integers or the integers modulo n, and let S be a finite nonempty subset of G. A *k-term arithmetic progression (k-AP)* in S is a set of distinct elements of the form

$$a, a+d, a+2d, \ldots, a+(k-1)d$$

where $d \geq 1$ and $k \geq 2$. An *r-coloring* of S is a function $c : S \to [r]$, where $[r] := \{1, \ldots, r\}$. We say such a coloring is *exact* if c is surjective. An arithmetic progression is called *rainbow* if the image of the progression under the r-coloring is injective. Formally, given $c : S \to [r]$ we say a k-term arithmetic progression is rainbow if $\{c(a + id) : i = 0, 1, \ldots, k-1\}$ has k distinct values.

The *anti-van der Waerden number* $\mathrm{aw}(S, k)$ is the smallest r such that every exact r-coloring of S contains a rainbow k-term arithmetic progression. Note that this tautologically defines $\mathrm{aw}(S, k) = |S| + 1$ whenever $|S| < k$, and this definition retains the property that there is a coloring with $\mathrm{aw}(S, k) - 1$ colors that has no rainbow k-AP. Since $\mathrm{aw}(S, 2) = 2$ for all S, we assume henceforth that $k \geq 3$. A preliminary study of the anti-van der Waerden number was done by Uherka in [13] and it should be noted the notation there is slightly different, with $AW(k, n)$ used to denote our $\mathrm{aw}([n], k)$. Other results have been obtained on colorings of the integers by Jungić, et al. [8] and on balanced colorings with no rainbow 3-AP by Axenovich and Fon-Der-Flaass [1] and Axenovich and Martin [2].

First, we consider the set $S = [n]$. The value of $\mathrm{aw}([n], 3)$ is logarithmic in n:

[1]Department of Mathematics, Iowa State University, Ames, IA 50011, USA.
{butler, craig, lhogben, kahogens, ljkramer, ricardo, chlin, rymartin, dstolee, warnberg, myoung}@iastate.edu
[2]American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306, USA (hogben@aimath.org).
[3]Research supported in part by National Security Agency grant H98230-13-1-0226.
[4]Department of Computer Science, Iowa State University, Ames, IA 50011, USA.
[5]Research supported in part by NSF DMS 0946431.
[6]Corresponding author.

Theorem 1.1. *For every integer $n \geq 9$,*

$$\lceil \log_3 n \rceil + 2 \leq \operatorname{aw}([n], 3) \leq \lceil \log_2 n \rceil + 1.$$

Moreover, $\operatorname{aw}([n], 3) = \lceil \log_3 n \rceil + 2$ *for* $n \in \{3, 4, 5, 6, 7\}$ *and* $\operatorname{aw}([8], 3) = 5$.

Theorem 1.1 is Proven by Lemmas 2.3 and 2.6 (for $n \geq 9$), and Remark 2.1 gives exact values of $\operatorname{aw}([n], 3)$ that justify the second statement. We conjecture that the lower bound is, essentially, correct:

Conjecture 1.2. *There exists a constant C such that* $\operatorname{aw}([n], 3) \leq \lceil \log_3 n \rceil + C$ *for all* $n \geq 3$.

The behavior of $\operatorname{aw}([n], k)$ is, however, different for $k \geq 4$. Instead of logarithmic, it is almost linear:

Theorem 1.3. *For $k \geq 4$,*

$$ne^{-O\left(\sqrt{\log n}\right)} < \operatorname{aw}([n], k) \leq ne^{-\log\log\log n - \omega(1)}.$$

Theorem 1.3 is established by Lemma 2.8 and Corollary 2.14.

Finally, we consider arithmetic progressions in the cyclic group $\mathbb{Z}_n$.

Remark 1.4. For positive integers n and k, $\operatorname{aw}(\mathbb{Z}_n, k) \leq \operatorname{aw}([n], k)$, because every AP in $[n]$ corresponds to an AP in $\mathbb{Z}_n$.

However, because progressions in $\mathbb{Z}_n$ may "wrap around," there are additional APs in $\mathbb{Z}_n$, some of which may be rainbow. Thus it is possible that a rainbow k-AP is required with a coloring of $\mathbb{Z}_n$ with $\operatorname{aw}([n], k) - 1$ colors, so strict inequality is possible. As we see in Theorem 1.5, there is an infinite sequence of values of n for which $\operatorname{aw}(\mathbb{Z}_n, 3) = 3$.

Theorem 1.5.

1. *For all positive integers m,* $\operatorname{aw}(\mathbb{Z}_{2^m}, 3) = 3$.
2. *For an integer $n \geq 2$ having every prime factor less than* 100,

$$\operatorname{aw}(\mathbb{Z}_n, 3) = 2 + f_2 + f_3 + 2f_4. \tag{1}$$

Here f_4 denotes the number of odd prime factors of n in the set $Q_4 := \{17, 31, 41, 43, 73, 89, 97\}$. The quantity f_3 is the number of odd prime factors of n in Q_3, where Q_3 is the set of all odd primes less than 100 *and not in Q_4. Both f_3 and f_4 are counted according to multiplicity. Finally, $f_2 = 0$ if n odd and $f_2 = 1$ if n is even.*

Theorem 1.5 is established by Lemma 3.2 and Corollary 3.15.

For $k \geq 4$, the bounds we obtain for $\operatorname{aw}(\mathbb{Z}_n, k)$ are the same as those for $\operatorname{aw}([n], k)$:

Theorem 1.6. *For $k \geq 4$,*

$$ne^{-O\left(\sqrt{\log n}\right)} < \operatorname{aw}(\mathbb{Z}_n, k) \leq ne^{-\log\log\log n - \omega(1)}.$$

Theorem 1.6 is established by Theorem 1.1, Remark 1.4, and Lemma 3.20.

The structure of the paper is as follows: Section 2 presents results pertaining to $\operatorname{aw}([n], k)$, with Theorem 1.1 proved in Section 2.1 and Theorem 1.3 proved in Section 2.2. Results pertaining to $\operatorname{aw}(\mathbb{Z}_n, k)$ appear in Section 3, with Theorem 1.5 proved in Section 3.1 and Theorem 1.6 proved in Section 3.2. Section 4 describes the methods and algorithms used to compute values of $\operatorname{aw}([n], k)$ and $\operatorname{aw}(\mathbb{Z}_n, k)$, while Section 5 contains conjectures and open questions for future research.

In the remainder of this section we establish a basic but necessary observation that $\operatorname{aw}(S, \cdot)$ is monotone in k.

Observation 1.7. *Let G be an additive (abelian) group such as the integers or the integers modulo n, let S be a finite nonempty subset of G, and let $k \geq 3$ be an integer. Then* $\operatorname{aw}(S, k) \leq \operatorname{aw}(S, k+1)$.

Observation 1.7 follows immediately from Proposition 1.8 below and was noted noted by Uherka in [13] for the function $\mathrm{aw}([n],\cdot)$.

Proposition 1.8. *Let G be an additive (abelian) group such as the integers or the integers modulo n, let S be a finite nonempty subset of G, and let $k \geq 3$ be an integer. If there is an exact r-coloring of S that has no rainbow k-AP then* $\mathrm{aw}(S,k) \geq r+1$.

Proof. Let c be an exact r-coloring of S with color set $\{1,\dots,r\}$ that has no rainbow k-AP. We proceed by constructing an exact $(r-1)$-coloring of S with no rainbow k-AP. For $x \in S$, define

$$\hat{c}(x) = \begin{cases} c(x) & \text{if} \quad c(x) \in \{1,\dots,r-2\}, \\ r-1 & \text{if} \quad c(x) \in \{r-1,r\}. \end{cases}$$

Note that $\hat{c}$ is an exact $(r-1)$-coloring of S. Let K be a k-AP in S. Since there is no rainbow k-AP under c there exists $j,\ell \in K$ such that $c(j)=c(\ell)$. It then follows that $\hat{c}(j)=\hat{c}(\ell)$. Hence K is not rainbow under the coloring $\hat{c}$. By the generality of K, $\hat{c}$ is an exact $(r-1)$-coloring of S that has no rainbow k-AP. Repeating this construction we obtain an exact $(r-i)$-coloring of S with no rainbow k-AP for $i \in \{1,2,\dots,r-1\}$. Therefore $\mathrm{aw}(S,k) \geq r+1$. □

2 $\mathrm{aw}([n],k)$

In this section we establish properties of $\mathrm{aw}([n],k)$. Sections 2.1 and 2.2 establish our main results for $\mathrm{aw}([n],3)$ and $\mathrm{aw}([n],k), k \geq 4$, respectively. Sections 2.3 and 2.4 contain additional results valid for all k and specific to $k=3$, respectively. Given an r-coloring c of $[n]$, the i^{th} *color class* is $C_i := \{x \in S : c(x) = i\}$.

In Table 1 we give our calculated values of $\mathrm{aw}([n],k)$ for $k \geq 3$. We have a larger list of known values in the case of $k=3$ that is included in Remark 2.1 below; in Table 1 we include only the values $\mathrm{aw}([n],3)$ for which we have a value for $\mathrm{aw}([n],4)$ so that we may compare them. We also restrict $n,k \geq 3$, and have stopped with $k = \lceil \frac{n}{2} \rceil + 1$, because $\mathrm{aw}([n],k) = n$ if and only if $k \geq \lceil \frac{n}{2} \rceil + 1$ (Proposition 2.16 below).

The growth rates when $k=3$ and when $k \geq 4$ appear to be different based on data given in Table 1. The upper bound of $\lceil \log_2 n \rceil + 1$ given in Proposition 2.6 for $k=3$ and the lower bound of $n^{1-o(1)}$ in 2.8 for $k \geq 4$ confirm that the growth rates are indeed radically different.

2.1 Main results for $\mathrm{aw}([n],3)$

Before we address Theorem 1.1, we show a summary of the computed data in this case in Remark 2.1 below.

Remark 2.1. The exact values of $\mathrm{aw}([n],3)$ are known from computer computations (described in Section 4) for $n \leq 58$, and are recorded here.

1. $\mathrm{aw}([n],3) = 2$ for $n \in \{1\}$.
2. $\mathrm{aw}([n],3) = 3$ for $n \in \{2,3\}$.
3. $\mathrm{aw}([n],3) = 4$ for $n \in \{4,\dots,7\} \cup \{9\}$.
4. $\mathrm{aw}([n],3) = 5$ for $n \in \{8\} \cup \{10,\dots,21\} \cup \{27\}$.
5. $\mathrm{aw}([n],3) = 6$ for $n \in \{22,\dots,26\} \cup \{28,\dots,58\}$.

Now we turn to the proof of Theorem 1.1.

$n \setminus k$	3	4	5	6	7	8	9	10	11	12	13	14
3	3											
4	4											
5	4	5										
6	4	6										
7	4	6	7									
8	5	6	8									
9	4	7	8	9								
10	5	8	9	10								
11	5	8	9	10	11							
12	5	8	10	11	12							
13	5	8	11	11	12	13						
14	5	8	11	12	13	14						
15	5	9	11	13	14	14	15					
16	5	9	12	13	15	15	16					
17	5	9	13	13	15	16	16	17				
18	5	10	14	14	16	17	17	18				
19	5	10	14	15	17	17	18	18	19			
20	5	10	14	16	17	18	19	19	20			
21	5	11	14	16	17	19	20	20	20	21		
22	6	12	14	17	18	20	21	21	21	22		
23	6	12	14	17	19	20	21	22	22	22	23	
24	6	12	15	18	20	20	22	23	23	23	24	
25	6	12	15	19	21	21	23	23	24	24	24	25

Table 1: Values of $\mathrm{aw}([n], k)$ for $3 \leq k \leq \frac{n+3}{2}$.

2.1.1 Theorem 1.1: Proof of lower bound

Proposition 2.2. *Let n be a positive integer and let $s \in \{-2, -1, 0, 1, 2\}$. Then* $\mathrm{aw}([3n-s], 3) \geq \mathrm{aw}([n], 3)+1$ *provided $n \geq s$.*

Proof. Let $r = \mathrm{aw}([n], 3)$ and $s \in \{0, 1, 2\}$. We construct an exact r-coloring of $[3n-s]$ that does not contain a rainbow 3-AP. By definition there exists an exact $(r-1)$-coloring, denoted c, of $[n]$ such that there is no rainbow 3-AP in $[n]$. Color $[3n-s]$ in the following manner: If $i+s$ is divisible by 3, then $\hat{c}(i) = c((i+s)/3)$, otherwise $\hat{c}(i) = r$. Consider a 3-AP, K, in $[3n-s]$. Then either the three terms in $K+s$ are all divisible by 3 or at least two of the terms in $K+s$ are not divisible by 3. If all terms in $K+s$ are divisible by 3, then K is not rainbow under $\hat{c}$, since there is no rainbow 3-AP under c. If two terms of $K+s$ are not divisible by 3 then those two terms are both colored r and K is not rainbow. Hence $\mathrm{aw}([3n-s], 3) \geq r+1$ for $s \in \{0, 1, 2\}$.

For $s \in \{-2, -1\}$, use the same coloring as for $s = 0$. □

Using Proposition 2.2, we establish the lower bound in Theorem 1.1.

Lemma 2.3. *Let n be a positive integer. Then* $\mathrm{aw}([n], 3) \geq \lceil \log_3 n \rceil + 2$.

Proof. The proof is by induction. The cases $n = 1, 2, 3$ are true by inspection. Suppose $n > 3$ and that $\mathrm{aw}([m], 3) \geq \lceil \log_3 m \rceil + 2$ for all m satisfying $1 \leq m < n$. We show that $\mathrm{aw}([n], 3) \geq \lceil \log_3 n \rceil + 2$. First, we write $n = 3m - s$, where $s \in \{0, 1, 2\}$ and $2 \leq m < n$. Then by Proposition 2.2,

$$\mathrm{aw}([n], 3) = \mathrm{aw}([3m-s], 3) \geq \mathrm{aw}([m], 3) + 1 \geq \lceil \log_3 m \rceil + 2 + 1 = \lceil \log_3(3m) \rceil + 2 \geq \lceil \log_3 n \rceil + 2. \quad \square$$

Example 2.4. Induction and the proof of Proposition 2.2 produce the following exact $(m+1)$-coloring of $[3^m]$ that does not have a rainbow 3-AP: For $x \in [3^m]$ with the prime factorization $x = 2^{e_2} 3^{e_3} 5^{e_5} \cdots p^{e_p}$, $c(x) = m + 1 - e_3$. This attains the value in Lemma 2.3.

2.1.2 Theorem 1.1: Proof of upper bound

Proposition 2.5. *For $n \geq 2$, there exists $m \leq \lfloor \frac{n}{2} \rfloor$ such that* $\mathrm{aw}([n], 3) \leq \mathrm{aw}([m], 3) + 1$.

Proof. We may assume that $n \geq 3$, since the case $n = 2$ follows by inspection. Let $r = \mathrm{aw}([n], 3)$. Then there exists an $(r-1)$-coloring, namely c, of $[n]$ that has no rainbow 3-AP. Let t be the length of a shortest consecutive integer sequence in $[n]$ that contains all $r-1$ colors, say the interval is $\{s+1, s+2, \ldots, s+t\}$ for some s. Define $\hat{c}$ to be an $(r-1)$-coloring of $[t] = \{1, 2, \ldots, t\}$ so that $\hat{c}(j) := c(s+j)$ for $1 \leq j \leq t$. Notice that $\hat{c}(1)$ and $\hat{c}(t)$ cannot be the same color and each must be the only element of its color class, or else we could find a smaller t. Let $\hat{c}(1) = a$ and define b_i to be the smallest element of $[t]$ such that $[b_i]$ has $i+1$ colors for $1 \leq i \leq r-2$. Note that if b_i is odd, i.e., $b_i = 2x+1$, then $\{1, x+1, 2x+1\}$ is a rainbow 3-AP. So the set of even numbers of $[t]$ are colored with exactly $r-2$ colors with no rainbow 3-AP. Define $m = \lfloor \frac{t}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor$ and consider the coloring $\tilde{c}$ of $[m]$ induced by the coloring $\hat{c}$ of the even integers in $[t]$. The coloring $\tilde{c}$ uses at least $r-2$ colors and has with no rainbow 3-AP, so $\mathrm{aw}([n], 3) - 1 = (r-2) + 1 \leq \mathrm{aw}([m], 3)$. □

Using Proposition 2.5, we establish the upper bound in Theorem 1.1.

Lemma 2.6. *For $n \geq 9$,* $\mathrm{aw}([n], 3) \leq \lceil \log_2 n \rceil + 1$

Proof. The proof is by strong induction on n. We consider $9 \leq n \leq 17$ as the base case, and the result is established for these values by computation (see Remark 2.1).

As the induction hypothesis, assume $\mathrm{aw}([m], 3) \leq \lceil \log_2 m \rceil + 1$ for $9 \leq m \leq n$, and suppose that $n \geq 17$. Then $9 \leq \lfloor \frac{n+1}{2} \rfloor \leq n$, and by Proposition 2.5, there exists an $m \leq \lfloor \frac{n+1}{2} \rfloor$ such that

$$\mathrm{aw}([n+1], 3) \leq \mathrm{aw}([m], 3) + 1.$$

So, by the induction hypothesis,

$$\begin{aligned} \mathrm{aw}([n+1], 3) &\leq \lceil \log_2 m \rceil + 2 \\ &\leq \left\lceil \log_2 \left\lfloor \frac{n+1}{2} \right\rfloor \right\rceil + 2 \\ &\leq \left\lceil \log_2 \left(\frac{n+1}{2} \right) \right\rceil + 2 \\ &= \lceil \log_2 (n+1) \rceil + 1. \end{aligned}$$

□

This completes the proof of Theorem 1.1.

2.2 Main results for $\mathrm{aw}([n], k), k \geq 4$

In this section we specialize to the case $k \geq 4$, focusing on lower and upper bounds that give $\mathrm{aw}([n], k) = n^{1-o(1)}$. Lemma 2.8 gives the lower bound and Corollary 2.14 gives the upper bound.

Let $\mathrm{sz}(n, k)$ denote the largest size of a set $S \subseteq [n]$ such that S contains no k-AP (similar notation was introduced in [5] in honor of Szemerédi [12]). Determining bounds on $\mathrm{sz}(n, k)$ is a fundamental problem in the study of arithmetic progressions. Behrend [3], Gowers [6], and others [9, 10] have established various bounds on $\mathrm{sz}(n, k)$. Proposition 2.7 provides a strong link between $\mathrm{sz}(n, k)$ and our anti-van der Waerden numbers, allowing us to use known results on $\mathrm{sz}(n, k)$ to bound $\mathrm{aw}(n, k)$.

Proposition 2.7. *For all $n > k \geq 3$,*

$$\mathrm{sz}(n, \lfloor k/2 \rfloor) + 1 \leq \mathrm{aw}([n], k) - 1 \leq \mathrm{sz}(n, k).$$

Proof. If c is an exact r-coloring of $[n]$ that contains no rainbow k-AP, then selecting one element of each color class creates a set S that contains no k-AP; therefore $\mathrm{aw}([n], k) - 1 \leq \mathrm{sz}(n, k)$. If S is a set in $[n]$ that contains no $\lfloor k/2 \rfloor$-AP, then color $[n]$ by giving each element in S a distinct color and the elements of $[n] \setminus S$ a new color. If a k-AP $\{a_1, a_2, \ldots, a_k\}$ is rainbow in this coloring, then exactly one such a_i is in $[n] \setminus S$. But this implies that the entries a_j where $j \not\equiv i \pmod 2$ form an AP in S with at least $\lfloor k/2 \rfloor$ terms, a contradiction. □

2.2.1 Theorem 1.3: Proof of lower bound

Lemma 2.6 and Behrend's results (stated in Theorem 2.10 and Proposition 2.11 below) show that the upper bound in Proposition 2.7 is not useful for $k = 3$. Observe that when $k \in \{4, 5\}$, the lower bound in Proposition 2.7 is trivial but is, in fact useful in the case of $k \geq 6$. We provide a similar lower bound for $k \in \{4, 5\}$ in Lemma 2.8 by carefully studying Behrend's original construction [3] of a relatively large set $S \subset [n]$ that contains no 3-AP, thus giving a lower bound on $\mathrm{sz}(n, 3)$.

Let $\{a_1, a_2, a_3, a_4\}$ be a 4-AP. A set $A \subset \{a_1, a_2, a_3, a_4\}$ of size $|A| = 3$ is called a *punctured 4-AP*. If such a punctured 4-AP A is not a 3-AP, then it is of the form $A = \{a_1, a_2, a_4\}$ or $A = \{a_1, a_3, a_4\}$. We prove that Behrend's construction in fact contains no punctured 4-AP (Proposition 2.9 below). This leads to Lemma 2.8 below.

Lemma 2.8. *There exists an absolute constant $b > 0$ such that for all $n, k \geq 4$,*

$$\mathrm{aw}([n], k) > ne^{-b\sqrt{\log n}} = n^{1-o(1)}.$$

The proof of Lemma 2.8 follows from Proposition 2.9, Theorem 2.10, Proposition 2.11 and Proposition 2.12, which follow.

Proposition 2.9. *Suppose $S \subseteq [n]$ does not contain any punctured 4-APs. Then* $\mathrm{aw}([n], k) > |S| + 1$ *for all* $n \geq k \geq 4$.

Proof. Color each member of S a distinct color, and color each integer in $[n] \setminus S$ with a new color called *zero*. If there is a rainbow 4-AP in this coloring, then at most one of the elements in this 4-AP is colored zero. Thus there must be a punctured 4-AP in the other colors, but S contains no punctured 4-AP. □

There is a bijection between vectors $\mathbf{x} = (x_1, \ldots, x_m)^\top \in \mathbb{Z}^m$ where $x_i \in \{0, 1, \ldots, 2d-2\}$ for all $i \in [m]$ and elements of $\{0, 1, \ldots, (2d-1)^m - 1\}$, by viewing $\mathbf{x}$ as a $(2d-1)$-ary representation of an integer:

$$\mathbf{x} = (x_1, \ldots, x_m)^\top \longleftrightarrow a_{\mathbf{x}} = \sum_{i=1}^{m} x_i (2d-1)^{i-1}.$$

Moreover, observe that if $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^m$ with $x_i, y_i \in \{0, \ldots, d-1\}, i = 1, \ldots, m$, are associated with $a_{\mathbf{x}}, a_{\mathbf{y}} \in \{0, 1, \ldots, (2d-1)^m - 1\}$ by this bijection, then $\mathbf{x} + \mathbf{y}$ has $x_i + y_i \in \{0, \ldots, 2d-1\}$ and $\mathbf{x} + \mathbf{y}$ is associated with $a_{\mathbf{x}+\mathbf{y}} = a_{\mathbf{x}} + a_{\mathbf{y}} \in \{0, 1, \ldots, (2d-1)^m - 1\}$.

Recall that for a vector $\mathbf{x} \in \mathbb{R}^m$, $||\mathbf{x}||^2 = \sum_{i=1}^m x_i^2$. Let m, ℓ, d be positive integers and define $X_\ell(m, d)$ to be the set of vectors $\mathbf{x} = (x_1, \ldots, x_m)^\top$ such that

1. $x_i \in \{0, \ldots, d-1\}$ for all $i \in \{1, \ldots, m\}$, and
2. $||\mathbf{x}||^2 = \ell$.

The set $S_\ell(m, d)$ of integers associated with the vectors in $X_\ell(m, d)$ via the map $\mathbf{x} \to a_{\mathbf{x}}$ forms a subset of integers in $\{0, 1, \ldots, (2d-1)^m - 1\}$. Behrend [3] used the pigeonhole principle to prove the following lemma; here we state the version from [4].

Theorem 2.10. [3, 4] *There exist absolute constants $b, b' > 0$ such that for all n and positive integers $m = m(n)$, $\ell = \ell(n)$, and $d = d(n)$ such that $S_\ell(m, d) \subseteq [n]$ and*

$$|S_\ell(m, d)| \geq \frac{b' n}{2^{\sqrt{8 \log_2 n}} (\log n)^{1/4}} \geq ne^{-b\sqrt{\log n}}.$$

The important property of $S_\ell(m, d)$ is that it avoids non-trivial arithmetic progressions. We include Behrend's simple proof of this fact for completeness.

Proposition 2.11. [3] *The set $S_\ell(m, d)$ contains no 3-AP.*

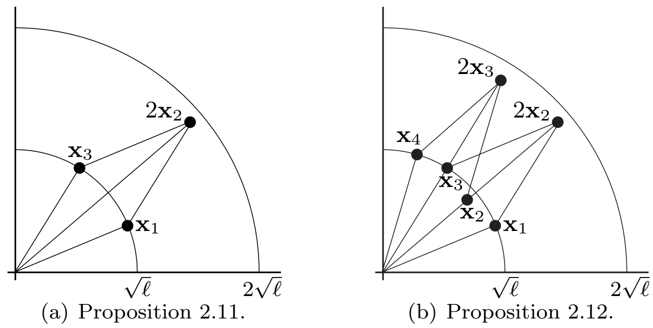

Figure 1: Proofs of Propositions 2.11 and 2.12.

Proof. Suppose $\{a_{\mathbf{x}_1}, a_{\mathbf{x}_2}, a_{\mathbf{x}_3}\}$ is a 3-AP in $S_\ell(m,d)$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be the associated vectors in $X_\ell(m,d)$. Since $a_{\mathbf{x}_1} + a_{\mathbf{x}_3} = 2a_{\mathbf{x}_2}$, we also have that $\mathbf{x}_1 + \mathbf{x}_3 = 2\mathbf{x}_2$. However, by the triangle inequality, we have that

$$2\sqrt{\ell} = 2||\mathbf{x}_2|| = ||\mathbf{x}_1 + \mathbf{x}_3|| \leq ||\mathbf{x}_1|| + ||\mathbf{x}_3|| = 2\sqrt{\ell},$$

and equality can only hold if $\mathbf{0}$, $\mathbf{x}_1$, $\mathbf{x}_3$ and $2\mathbf{x}_2$ are collinear. However, since $||\mathbf{x}_1|| = ||\mathbf{x}_3||$, this would imply $\mathbf{x}_1 = \mathbf{x}_3$ and thus $a_{\mathbf{x}_1} = a_{\mathbf{x}_3}$, a contradiction. □

Proposition 2.12. *The set $S_\ell(m,d)$ contains no punctured 4-AP.*

Proof. Let $\{a_{\mathbf{x}_1}, a_{\mathbf{x}_2}, a_{\mathbf{x}_3}, a_{\mathbf{x}_4}\}$ be a 4-AP. Since $S_\ell(m,d)$ contains no 3-AP, it must be that one of $a_{\mathbf{x}_2}$ or $a_{\mathbf{x}_3}$ is not in $S_\ell(m,d)$. Assume by symmetry that $a_{\mathbf{x}_2} \notin S_\ell(m,d)$ and $a_{\mathbf{x}_1}, a_{\mathbf{x}_3}, a_{\mathbf{x}_4} \in S_\ell(m,d)$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ be the associated vectors where $\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4 \in X_\ell(m,d)$.

Since $a_{\mathbf{x}_1} + a_{\mathbf{x}_3} = 2a_{\mathbf{x}_2}$, we have $\mathbf{x}_1 + \mathbf{x}_3 = 2\mathbf{x}_2$. However, as in the proof of Proposition 2.11, this implies that $||\mathbf{x}_2|| < \sqrt{\ell}$. Since $a_{\mathbf{x}_2} + a_{\mathbf{x}_4} = 2a_{\mathbf{x}_3}$, we have $\mathbf{x}_2 + \mathbf{x}_4 = 2\mathbf{x}_3$. However, this implies that

$$2\sqrt{\ell} = 2||\mathbf{x}_3|| = ||\mathbf{x}_2 + \mathbf{x}_4|| \leq ||\mathbf{x}_2|| + ||\mathbf{x}_4|| < 2\sqrt{\ell},$$

a contradiction. □

Lemma 2.8 now follows by combining Propositions 2.9 and 2.12. It may be possible that the bound in Lemma 2.8 could be improved by using the construction of Elkin [4, 7] that avoids 3-APs using $\frac{bn(\log n)^{1/4}}{2^{\sqrt{8\log_2 n}}}$ elements for some constant $b > 0$. An immediate use of the coloring in Lemma 2.8 with such a set demonstrates $\mathrm{aw}([n],k) > \frac{bn(\log n)^{1/4}}{2^{\sqrt{8\log_2 n}}}$ for all $k \geq 6$. Further use of constructions of Rankin [10] or Laba and Lacey [9] of large sets that avoid k-APs could slightly improve the asymptotics of $\mathrm{aw}([n],k)$, but these bounds are all of the form $n^{1-o(1)}$.

2.2.2 Theorem 1.3: Proof of upper bound

A theorem of Gowers, stated here as Theorem 2.13, provides an upper bound for $\mathrm{aw}([n],k)$. However, n must be very large compared to k for this upper bound to be significantly different than the naive upper bound of n itself.

Theorem 2.13. [6, Theorem 1.3] *For every positive integer k there is a constant $b = b(k) > 0$ such that every subset of $[n]$ of size at least $n(\log_2\log_2 n)^{-b}$ contains a k-AP. Moreover, b can be taken to be $2^{-2^{k+9}}$.*

Corollary 2.14. *Let n and k be positive integers. Then there exists a constant b such that $\mathrm{aw}([n],k) \leq \lceil n(\log_2\log_2 n)^{-b}\rceil$. That is, for a fixed positive integer k, the function $\mathrm{aw}([n],k)$ of n is $o(\frac{n}{\log\log n})$.*

Proof. Consider an exact t-coloring of $[n]$, where $t := \lceil n(\log_2\log_2 n)^{-b}\rceil$ and $b = 2^{-2^{k+9}}$. Since the coloring is exact, there exists a set $A \subseteq [n]$ of t differently colored integers. By Theorem 2.13, A contains a k-AP. Therefore $\mathrm{aw}([n],k) \leq t$. □

Note that the upper bound in Corollary 2.14 can be expressed as $ne^{-\log\log\log n - \omega(1)}$. Then combining this upper bound on $\mathrm{aw}([n],k)$ and the lower bound from Lemma 2.8, we have that for $k \geq 4$

$$ne^{-b\sqrt{\log n}} < \mathrm{aw}([n],k) \leq ne^{-\log\log\log n - \omega(1)}.$$

This completes the proof of Theorem 1.3.

2.3 Additional results for $\mathrm{aw}([n],k)$ valid for all k

In this section we present some additional elementary results for $\mathrm{aw}([n],k)$. The next proposition describes a relationship between $\mathrm{aw}([n],k)$ and $\mathrm{aw}([n-1],k)$.

Proposition 2.15. *Let n and k be positive integers. Then* $\mathrm{aw}([n],k) \leq \mathrm{aw}([n-1],k) + 1$.

Proof. Let $r = \mathrm{aw}([n],k)$. Note that if $n < k$ our result follows from the definition. Suppose $n \geq k$. Then there is some exact $(r-1)$-coloring of $[n]$ that has no rainbow k-AP, and without loss of generality n is colored $r-1$. Consider this coloring restricted to $[n-1]$. Then we have two cases:

1. This is an exact $(r-1)$-coloring of $[n-1]$.

2. The only integer in $[n]$ with the color $r-1$ is n, so this is an exact $(r-2)$-coloring of $[n-1]$.

Note that since $[n]$ had no rainbow k-AP in both of our cases we still do not have a rainbow k-AP. So by Proposition 1.8 we have $\mathrm{aw}([n-1],k) \geq r-1 = \mathrm{aw}([n],k) - 1$ and the result follows. ☐

In the next proposition we characterize the values of k for which $\mathrm{aw}([n],k) = n$.

Proposition 2.16. *Let n and k be positive integers with $k \leq n$. Then $\mathrm{aw}([n],k) = n$ if and only if $k \geq \lceil \frac{n}{2} \rceil + 1$.*

Proof. Suppose $k \geq \lceil \frac{n}{2} \rceil + 1$. We show that $\mathrm{aw}([n],k) > n-1$. Color $\lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} \rceil + 1$ with the same color and all the remaining integers with unique colors. This is an exact $(n-1)$-coloring. Since $k \geq \lceil \frac{n}{2} \rceil + 1$, the integers in any k-AP must be consecutive integers, and the values $\lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} \rceil + 1$ must be contained in any k-AP. Hence no k-AP is rainbow.

For the converse, suppose $\mathrm{aw}([n],k) = n$. Color $[n]$ with $n-1$ colors such that there is no rainbow k-AP. Therefore exactly one color class has size two and the rest have size one. Denote the color class of size two as $C = \{n_1, n_2\}$, $n_1 < n_2$. Then every k-AP contains both n_1 and n_2, or else we would have a rainbow k-AP. Suppose that $k \leq \lceil \frac{n}{2} \rceil$. Then $\{1, 2, \ldots, k\}$ and $\{n-k+1, n-k+2, \ldots, n\}$ are k-APs. Note that $\{1, 2, \ldots, k\} \subseteq \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$ and $\{n-k+1, n-k+2, \ldots, n\} \subseteq \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$. Then $n_1, n_2 \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\} \cap \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$. This intersection is empty or contains one element depending on whether n is even or odd. In both cases, this contradicts the fact that $n_1 \neq n_2$ and $n_1, n_2 \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\} \cap \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$. Therefore $k \geq \lceil \frac{n}{2} \rceil + 1$. ☐

The following upper bound was proved by Uherka [13].

Proposition 2.17. [13] *Let n, k, n_1, and n_2 be positive integers such that $k \leq n_1 \leq n_2 \leq n$ and $n_1 + n_2 = n$. Then* $\mathrm{aw}([n],k) \leq \mathrm{aw}([n_1],k) + \mathrm{aw}([n_2],k) - 1$.

2.4 Additional results for $\mathrm{aw}([n],3)$

In this section we establish additional bounds on $\mathrm{aw}([n],3)$ in Lemma 2.18 and Corollary 2.19, and use Corollary 2.19 together with Remark 2.1 and Proposition 2.2 and Lemma 2.3 to compute (at least) 93 additional exact values for $\mathrm{aw}([n],3)$.

Lemma 2.18. *Let c be an exact r-coloring of $[n]$ that does not have a rainbow 3-AP. For $i \in [r]$, define $b_i \in [n]$ to be the least x such that the induced coloring on $[x]$ has exactly i colors. Then for all $i \in [r-1]$, $b_{i+1} \geq 2b_i$. Furthermore, for any $1 \leq i \leq j \leq r$, we have $b_j \geq 2^{j-i} b_i$.*

Proof. Observe that $b_1 = 1$. Suppose that $b_{i+1} < 2b_i$ for some $i \in [r-1]$. Then $\{2b_i - b_{i+1}, b_i, b_{i+1}\}$ is a rainbow 3-AP. The last statement follows by induction, since $b_i \le 2^{-1}b_{i+1} \le 2^{-2}b_{i+2} \le \cdots \le 2^{-(j-i)}b_j$. □

Corollary 2.19. *Let* m, n, *and* ℓ *be positive integers. If* $m < n < 2^\ell(m+1)$, *then* $\mathrm{aw}([n],3) \le \mathrm{aw}([m],3)+\ell$.

Proof. Suppose not. Then there exists $m, \ell \ge 1$ and n with $m < n < 2^\ell(m+1)$ such that there is a coloring c on $[n]$ using exactly $r = \mathrm{aw}([m],3)+\ell$ colors that does not have a rainbow 3-AP. For $i \in [r]$, let $b_i \in [n]$ be the least x such that the induced coloring on $[x]$ has exactly i colors. Since $r - \ell = \mathrm{aw}([m],3)$, we must have $b_{r-\ell} \ge m+1$, since otherwise the induced coloring on $[m]$ contains at least $\mathrm{aw}([m],3)$ colors, which is impossible. Thus by Lemma 2.18, $n \ge b_r \ge 2^\ell b_{r-\ell} \ge 2^\ell(m+1)$, which contradicts our assumption on n. □

Corollary 2.20. $\mathrm{aw}([n],3) = 7$ *for* $64 \le n \le 80$.

Proof. Since $\mathrm{aw}([m],3) = 6$ for $22 \le m \le 26$, and $3\cdot 22-2 = 64$ and $3\cdot 26+2 = 80$, we see that $\mathrm{aw}([n],3) \ge 7$, by Proposition 2.2. Since $27 < 64 \le n \le 80 < 112 = 4\cdot 28$, we have $\mathrm{aw}([n],3) \le \mathrm{aw}([27],3)+2 = 5+2 = 7$ by Corollary 2.19 and Remark 2.1. □

Corollary 2.21. $\mathrm{aw}([n],3) = 7$ *for* $82 \le n \le 111$.

Proof. Since $27 < 82 \le n \le 111 < 112 = 4\cdot 28$, we have $\mathrm{aw}([n],3) \le \mathrm{aw}([27],3)+2 = 5+2 = 7$ by Corollary 2.19 and Remark 2.1. Also, since $3^4 = 81 < n \le 111 \le 243 = 3^5$, we have $4 < \log_3 n \le 5$, so that $\mathrm{aw}([n],3) \ge \lceil \log_3 n \rceil + 2 = 5+2 = 7$ by Lemma 2.3. □

Corollary 2.22. $\mathrm{aw}([n],3) = 8$ *for* $190 \le n \le 235$.

Proof. Since $\mathrm{aw}([m],3) = 7$ for $64 \le m \le 80$, and $3\cdot 64-2 = 190$ and $3\cdot 80+2 = 242$, we see that $\mathrm{aw}([n],3) \ge 8$ for $190 \le n \le 242$, by Proposition 2.2. For $58 < n < 236 = 2^2\cdot(58+1)$, we see that $\mathrm{aw}([n],3) \le \mathrm{aw}([58],3)+2 = 6+2 = 8$, by Corollary 2.19 and Remark 2.1. □

Finally we combine the upper and lower bounds.

Proposition 2.23. *If* $3^u < n < 2\cdot 3^u+2$, *then* $u+3 \le \mathrm{aw}([n],3) \le \mathrm{aw}([3^u],3)+1$. *If* $2\cdot 3^u+1 < n < 4\cdot 3^u+4$, *then* $u+3 \le \mathrm{aw}([n],3) \le \mathrm{aw}([3^u],3)+2$.

Proof. The lower bounds follow immediately from Lemma 2.3, and the first upper bound follows immediately from Corollary 2.19. For the second, apply Corollary 2.19 when $m = 2\cdot 3^u+1$ to obtain $\mathrm{aw}([n],3) \le \mathrm{aw}([m],3)+1$ and since $3^u < m < 2\cdot 3^u+2$, $\mathrm{aw}([m],3) < \mathrm{aw}([3^u],3)+2$. □

3 $\mathrm{aw}(\mathbb{Z}_n,k)$

In this section we establish properties of $\mathrm{aw}(\mathbb{Z}_n,k)$. Sections 3.1 and 3.2 establish our main results for $\mathrm{aw}(\mathbb{Z}_n,3)$ and $\mathrm{aw}(\mathbb{Z}_n,k), k \ge 4$, respectively. Section 3.3 contains additional results.

Please note that for $x \in \mathbb{Z}$, we will also use x to denote the equivalence class $\{x+in : i \in \mathbb{Z}\}$ in $\mathbb{Z}_n$. Because arithmetic progressions may "wrap around" in the group $\mathbb{Z}_n$, we call attention to the fact that we consider only k-APs that include k distinct members of $\mathbb{Z}_n$. Naturally, one of our first questions about $\mathrm{aw}(\mathbb{Z}_n,k)$ concerns its relationship with $\mathrm{aw}([n],k)$. Lemma 3.2 below and Lemma 2.3 show that $\mathrm{aw}(\mathbb{Z}_n,k)$ need not be asymptotic to $\mathrm{aw}([n],k)$ for $k = 3$ and $n = 2^m$. However, we do have the simple bound $\mathrm{aw}(\mathbb{Z}_n,k) \le \mathrm{aw}([n],k)$ (already stated in Remark 1.4).

	0	1	2	3	4	5	6	7	8	9
0–9				3	3	3	4	3	3	4
10–19	4	3	4	3	4	4	3	4	5	3
20–29	4	4	4	3	4	4	4	5	4	3
30–39	5	4	3	4	5	4	5	3	4	4
40–49	4	4	5	4	4	5	4	3	4	4
50–59	5	5	4	3	6	4	4	4	4	3
60–69	5	3	5	5	3	4	5	3	5	4
70–79	5	3	5	4	4	5	4	4	5	3
80–89	4	6	5	3	5	5	5	4	4	4
90–99	6	4	4	5	4	4	4	4	5	5

Table 2: Computed values of $\mathrm{aw}(\mathbb{Z}_n, 3)$ for $n = 3, \ldots, 99$ (the row label gives the range of n and the column heading is the ones digit within this range).

3.1 Main results for $\mathrm{aw}(\mathbb{Z}_n, 3)$

When we turn to the special case $k = 3$, many values of $\mathrm{aw}(\mathbb{Z}_n, 3)$ can be computed, and new phenomena appear. Our main results in this case are described by Theorem 1.5, which we establish here.

Currently available computational data is given in Table 2; the row label displays the range of n for which the values of $\mathrm{aw}(\mathbb{Z}_n, 3)$ are reported in that row, and the column heading is the ones digit within this range. This data led to the discovery of several results.

Definition 3.1. When dealing with a coloring c of $\mathbb{Z}_{st}$, the i^{th} *residue class* modulo s is $R_i := \{j \in \mathbb{Z}_{st} : j \equiv i \pmod{s}\}$ and the i^{th} *residue palette* modulo s is $P_i := \{c(\ell) : \ell \in R_i\}$. For a positive integer t, we call the elements of the two residue classes, R_0 and R_1, modulo 2 in $\mathbb{Z}_{2t}$ the *even numbers* and the *odd numbers*, respectively.

3.1.1 Proof of Theorem 1.5, $\mathrm{aw}(\mathbb{Z}_{2^m}, 3)$

As we see from the computed values in Table 2, $\mathrm{aw}(\mathbb{Z}_n, 3)$ does not have an increasing lower bound, and in fact $\mathrm{aw}(\mathbb{Z}_n, 3) = 3$ whenever n is a power of 2, as proved in Lemma 3.2 below, which is the first part of Theorem 1.5.

Lemma 3.2. *For all positive integers m,* $\mathrm{aw}(\mathbb{Z}_{2^m}, 3) = 3$.

Proof. We prove this by induction on $m \geq 1$ and observe that $\mathrm{aw}(\mathbb{Z}_2, 3) = 3$ by definition and $\mathrm{aw}(\mathbb{Z}_{2^m}, 3) \geq 3$ trivially. Let $m > 1$ and let c be a coloring of $\mathbb{Z}_{2^m}$ with no rainbow 3-AP. We will show that the number of colors in c is at most 2.

Let A and B denote the residue palettes of the even and odd numbers. Since the even numbers and the odd numbers each form a copy of $\mathbb{Z}_{2^{m-1}}$ within $\mathbb{Z}_{2^m}$, the fact that $\mathrm{aw}(\mathbb{Z}_{2^{m-1}}, 3) = 3$ implies that $|A| \leq 2$ and $|B| \leq 2$. If $A \subseteq B$ or $B \subseteq A$, then at most two colors are used by c and the result follows.

So suppose $a \in A \setminus B$ and $b \in B \setminus A$. There exist elements $\ell_a, \ell_b \in \mathbb{Z}_{2^m}$ such that $c(\ell_a) = a$ and $c(\ell_b) = b$. Observe that ℓ_a is even and ℓ_b is odd, so for $d := \ell_b - \ell_a$, d is an odd number. Since $\gcd(d, 2^m) = 1$, d is a generator of $\mathbb{Z}_{2^m}$ as an additive group. Thus, every element in $\mathbb{Z}_{2^m}$ is describable as an element $\ell_a + gd$ for some $g \in \{0, \ldots, 2^m - 1\}$.

We will prove that

$$c(\ell_a + gd) = \begin{cases} a & \text{if } g \text{ is even,} \\ b & \text{if } g \text{ is odd,} \end{cases}$$

by induction on $g \geq 0$. Observe that $c(\ell_a) = a$ and $c(\ell_a + d) = c(\ell_b) = b$, so this holds for $g \in \{0, 1\}$. Let $g > 1$ and consider the 3-AP $K := \{\ell_a + (g-2)d, \ell_a + (g-1)d, \ell_a + gd\}$, which is not rainbow.

If g is even, then $c(\ell_a + (g-2)d) = a$ and $c(\ell_a + (g-1)d) = b$ by the induction hypothesis. Thus, since the 3-AP, K, is not rainbow, $c(\ell_a + gd) \in \{a, b\}$. But since $b \notin A$, we have $c(\ell_a + gd) = a$. If g is odd, then $c(\ell_a + (g-2)d) = b$ and $c(\ell_a + (g-1)d) = a$. Thus, since the 3-AP, K, is not rainbow, $c(\ell_a + gd) \in \{a, b\}$. But since $a \notin B$, we have $c(\ell_a + gd) = b$. Thus, all elements of $\mathbb{Z}_{2^m}$ are colored with either a or b, so only two colors were used. Hence $\mathrm{aw}(\mathbb{Z}_{2^m}, 3) = 3$. $\square$

3.1.2 Proof of Theorem 1.5, general $\mathrm{aw}(\mathbb{Z}_n, 3)$

Next we present a series of results that lead to lower and upper bounds on $\mathrm{aw}(\mathbb{Z}_n, 3)$ in terms of the prime factorization of n. In many cases these bounds agree, and so $\mathrm{aw}(\mathbb{Z}_n, 3)$ is determined for arbitrarily large n such that all prime factors are less than 100. Many odd primes p have $\mathrm{aw}(\mathbb{Z}_p, 3) = 3$ (see Table 2 above). However, there are several examples of odd primes p for which $\mathrm{aw}(\mathbb{Z}_p, 3) = 4$. We show in Example 3.3 below an explicit exact coloring that establishes $\mathrm{aw}(\mathbb{Z}_{17}, 3) \geq 4$.

Example 3.3. Coloring the elements of $\mathbb{Z}_{17}$ in order as

$$3\ 1\ 1\ 2\ 1\ 2\ 2\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1$$

is an exact 3-coloring that does not contain a rainbow 3-AP. Computations establish that equality holds and so $\mathrm{aw}(\mathbb{Z}_{17}, 3) = 4$ (see Table 2 above).

Proposition 3.4. *For s odd, $s \geq 3$, and $t \geq 2$, $\mathrm{aw}(\mathbb{Z}_{st}, 3) \geq \mathrm{aw}(\mathbb{Z}_t, 3) + 1$.*

Proof. We exhibit a coloring $\hat{c}$ of $\mathbb{Z}_{st}$ with $\mathrm{aw}(\mathbb{Z}_t, 3)$ colors and no rainbow 3-AP, thus establishing that $\mathrm{aw}(\mathbb{Z}_{st}, 3) \geq \mathrm{aw}(\mathbb{Z}_t, 3) + 1$. Let $r := \mathrm{aw}(\mathbb{Z}_t, 3)$. There exists a coloring c of $\mathbb{Z}_t$ using $r - 1$ colors that has no rainbow 3-AP. Consider the residue classes and residue palettes modulo s. Use the coloring c to color $R_0 = \{0, s, 2s, \ldots, (t-1)s\}$ by defining $\hat{c}(is) := c(i)$, and for all of the remaining elements ℓ of $\mathbb{Z}_{st}$, $\hat{c}(\ell) = r$. We show that there is no rainbow 3-AP in $\mathbb{Z}_{st}$. Consider any 3-AP in $\mathbb{Z}_{st}$. If at most one of the terms is in R_0, then at least two of the terms are not in R_0 and so are the same color, implying the 3-AP is not rainbow. If all three terms are in R_0, then again the 3-AP is not rainbow, because a rainbow 3-AP in R_0 would necessarily arise from a (nonexistent) rainbow 3-AP in the coloring c of $\mathbb{Z}_t$. If two of the terms are in R_0, we show that the third term must also be in R_0. This is immediate if $a+d \in R_0$. Suppose $a, a+2d \in R_0$, so $2d \equiv 0 \pmod{s}$. Since 2 and s are relatively prime, $d \equiv 0 \pmod{s}$ and $a + d \in R_0$. Therefore, we have found an r-coloring of $\mathbb{Z}_{st}$ with no rainbow 3-AP, so $\mathrm{aw}(\mathbb{Z}_{st}, 3) \geq r + 1 = \mathrm{aw}(\mathbb{Z}_t, 3) + 1$. □

The next result gives our main recursive upper bound for $\mathrm{aw}(\mathbb{Z}_n, 3)$.

Proposition 3.5. *Suppose s is odd, and either t is odd or $t = 2^m$. Then*

$$\mathrm{aw}(\mathbb{Z}_{st}, 3) \leq \mathrm{aw}(\mathbb{Z}_s, 3) + \mathrm{aw}(\mathbb{Z}_t, 3) - 2.$$

Proposition 3.5 is established by Propositions 3.8 (t odd) and 3.11 ($t = 2^m$) below, after the proofs of necessary lemmas, but we first point out an immediate corollary to Proposition 3.5 together with Proposition 3.4.

Corollary 3.6. *Suppose p is an odd prime such that $\mathrm{aw}(\mathbb{Z}_p, 3) = 3$. Then $\mathrm{aw}(\mathbb{Z}_{p^m}, 3) = m + 2$.*

Examples of primes p to which Corollary 3.6 applies include 3, 5, 7, 11, 13, and 19; additional primes may be found in Table 2. Next we prove a technical lemma used in the proof of Proposition 3.8 and elsewhere.

Proposition 3.7. *Let s be an odd positive integer. Suppose c is a coloring of $\mathbb{Z}_{st}$ that does not have a rainbow 3-AP. Let $R_0, R_1, \ldots, R_{s-1}$ be the residue classes modulo s in $\mathbb{Z}_{st}$ with associated residue palettes P_i. Let m be an index such that $|P_m| \geq |P_i|$ for all i. Then $|P_i \setminus P_m| \leq 1$ for all i.*

Proof. For arbitrary nonnegative integers h and j, we show that $|P_{h+j} \setminus P_h| \geq 2$ implies $P_h = P_{h+2j}$. Assume $|P_{h+j} \setminus P_h| \geq 2$. Suppose first that $P_{h+2j} \setminus P_h$ is not empty and $z \in P_{h+2j} \setminus P_h$. Since $|P_{h+j} \setminus P_h| \geq 2$, we can pick some $y \in P_{h+j} \setminus P_h$ other than z. Let $\ell_y, \ell_z \in \mathbb{Z}_{st}$ with $\ell_y \in R_{h+j}, \ell_z \in R_{h+2j}$ and $c(\ell_y) = y$, $c(\ell_z) = z$. Define $\ell_x := 2\ell_y - \ell_z \in R_h$, so $x := c(\ell_x)$ is a color in P_h. By the choice of y, $y \neq z$; $z \neq x$ since $z \in P_{h+2h} \setminus P_h$ and $x \in P_h$; $x \neq y$ since $y \in P_{h+j} \setminus P_h$ and $x \in P_h$. Thus ℓ_x, ℓ_y, ℓ_z is a rainbow 3-AP, a contradiction. Therefore we conclude that $P_{h+2j} \subseteq P_h$. With this condition, we consider the case $P_h \setminus P_{h+2j}$ is not empty. Let $x \in P_h \setminus P_{h+2j}$. Similarly, it is possible to pick $y \in P_{h+j} \setminus P_h$. Let $\ell_x, \ell_y \in \mathbb{Z}_{st}$ with $\ell_x \in R_h$, $\ell_y \in R_{h+j}$, and $c(\ell_x) = x$, $c(\ell_y) = y$. Thus $\ell_z := 2\ell_y - \ell_x \in R_{h+2j}$ and so $z := c(\ell_z)$ is a color in P_{h+2j}. Again, $x \neq y$ by the choice of y; $x \neq z$ since $x \in P_h \setminus P_{h+2j}$ and $z \in P_{h+2h}$; $y \neq z$ since $y \in P_{h+j} \setminus P_h$ and $z \in P_{h+2j} \subseteq P_h$. Since we again have a contradiction, $P_h = P_{h+2j}$.

Next we show that $|P_{h+j} \setminus P_h| \geq 2$ implies $|P_h \setminus P_{h+j}| \leq 1$. Suppose $|P_{h+j} \setminus P_h| \geq 2$ and $|P_h \setminus P_{h+j}| \geq 2$, and then obtain a contradiction. By the result just established, $P_h = P_{h+2j}$. Since $|P_{h+2j} \setminus P_{h+j}| = |P_h \setminus P_{h+j}| \geq 2$, $P_{h+j} = P_{h+3j}$. Therefore $P_h = P_{h+qj}$ whenever q is even and $P_{h+j} = P_{h+qj}$ whenever q is odd. Since s is odd, the order d of j in $\mathbb{Z}_s$ is also an odd number. That means $P_h = P_{h+dj} = P_{h+j}$, which is a contradiction.

Finally, since $|P_m|$ is chosen to be maximum, $|P_m \setminus P_j| \geq 2$ whenever $|P_j \setminus P_m| \geq 2$, which is impossible. Hence $|P_j \setminus P_m| \leq 1$. $\square$

Proposition 3.8. *Suppose s and t are both odd. Then* $\mathrm{aw}(\mathbb{Z}_{st}, 3) \leq \mathrm{aw}(\mathbb{Z}_s, 3) + \mathrm{aw}(\mathbb{Z}_t, 3) - 2$.

Proof. Let c be a coloring of $\mathbb{Z}_{st}$ that does not have a rainbow 3-AP. Consider the residue classes and residue palettes modulo s and without loss of generality assume $|P_0| \geq |P_i|$ for all i. We claim that

$$\left|\bigcup_{i=0}^{s-1} P_i\right| \leq (\mathrm{aw}(\mathbb{Z}_s, 3) - 1) + (\mathrm{aw}(\mathbb{Z}_t, 3) - 1) - 1. \tag{2}$$

The proof is by contradiction. Assume that (2) is false, i.e., assume

$$\left|\bigcup_{i=0}^{s-1} P_i\right| \geq (\mathrm{aw}(\mathbb{Z}_s, 3) - 1) + (\mathrm{aw}(\mathbb{Z}_t, 3) - 1) \tag{3}$$

and define a coloring $\hat{c}$ of $\mathbb{Z}_s = \{0, 1, \ldots, s-1\}$ in the following way: Let α be a color not in $\bigcup_{i=1}^{s-1}(P_i \setminus P_0)$ and define

$$\hat{c}(i) = \begin{cases} \alpha & \text{if } P_i \subseteq P_0, \\ \text{the element of } P_i \setminus P_0 & \text{if } P_i \not\subseteq P_0. \end{cases}$$

Note that Proposition 3.7 implies that the required element in $P_i \setminus P_0$ is unique, so this coloring is well-defined. Since c does not have a rainbow 3-AP, we know $|P_0| \leq \mathrm{aw}(\mathbb{Z}_t, 3) - 1$ so

$$\left|\bigcup_{i=1}^{s-1}(P_i \setminus P_0)\right| \geq \left|\bigcup_{i=0}^{s-1} P_i\right| - (\mathrm{aw}(\mathbb{Z}_t, 3) - 1) \geq (\mathrm{aw}(\mathbb{Z}_s, 3) - 1) + (\mathrm{aw}(\mathbb{Z}_t, 3) - 1) - (\mathrm{aw}(\mathbb{Z}_t, 3) - 1) = \mathrm{aw}(\mathbb{Z}_s, 3) - 1.$$

Note that every color that is not in P_0, together with α, is used in $\hat{c}$, so $\hat{c}$ uses at least $\mathrm{aw}(\mathbb{Z}_s, 3)$ colors. Thus a rainbow 3-AP exists in $\hat{c}$.

We show that a rainbow 3-AP in $\hat{c}$ implies a rainbow 3-AP in c, providing a contradiction and establishing that (2) is true. Let x, y, z be a rainbow 3-AP in $\mathbb{Z}_s$ using coloring $\hat{c}$, with $y = x + d \pmod{s}$ and $z = x + 2d \pmod{s}$. Since x, y, z is rainbow, $\hat{c}(u) \neq \hat{c}(v)$ for all distinct $u, v \in \{x, y, z\}$, and so at most one $u \in \{x, y, z\}$ has $\hat{c}(u) = \alpha$. Note that by definition $\hat{c}(u) \in P_u$ or $\hat{c}(u) = \alpha$ for $u \in \{x, y, z\}$.

Case 1: $\hat{c}(z) \neq \alpha$ and $\hat{c}(y) \neq \alpha$. Then we can find g_2 and g_3 such that $c(g_2 s + y) = \hat{c}(y)$ and $c(g_3 s + z) = \hat{c}(z)$.
Define $d' := (g_3 s + z) - (g_2 s + y)$. Then

$$\begin{aligned} (g_3 s + z) - d' &= (g_2 s + y) \equiv y \pmod{s} \\ (g_3 s + z) - 2d' &= 2g_2 s + 2y - g_3 s - z \equiv 2y - z \equiv 2(x + d) - (x + 2d) \equiv x \pmod{s}. \end{aligned}$$

With $\ell := (g_3 s + z) - 2d'$, consider the 3-AP $\{\ell, (g_3 s + z) - d', (g_3 s + z)\}$. We show that this 3-AP is rainbow: Note that $\hat{c}(y) \notin P_0$ and $\hat{c}(z) \notin P_0$. If $\hat{c}(x) = \alpha$, then $P_x \subseteq P_0$, so $\ell \in R_x$ implies $c(\ell) \neq \hat{c}(y) = c(g_2 s + y)$ and $c(\ell) \neq \hat{c}(z) = c(g_3 s + z)$. If $\hat{c}(x) \neq \alpha$, then $\hat{c}(x)$ is the unique element of $P_x \setminus P_0$ and $\hat{c}(x) \neq \hat{c}(y), \hat{c}(z)$, so $\ell \in R_x$ implies $c(\ell) \neq \hat{c}(y)$ and $c(\ell) \neq \hat{c}(z)$. Thus c has a rainbow 3-AP, contradicting our assumption (3). The case where both $\hat{c}(x) \neq \alpha$ and $\hat{c}(y) \neq \alpha$ is symmetric to Case 1. So only Case 2 remains.

Case 2: $\hat{c}(y) = \alpha$. Then $\hat{c}(x) \in P_x \setminus P_0$ and $\hat{c}(z) \in P_z \setminus P_0$, so we can find g_1 and g_3 such that $c(g_1 s + x) = \hat{c}(x)$ and $c(g_3 s + z) = \hat{c}(z)$, and define $e := (g_3 s + z) - (g_1 s + x)$. Since st is odd, 2 is invertible in Z_{st} and there exists d' such that $2d' \equiv e \pmod{st}$, and hence $2d' \equiv e \pmod{s}$. Also, $e \equiv z - x \equiv 2d \pmod{s}$. Thus $2d \equiv 2d' \pmod{s}$ and so $d \equiv d' \pmod{s}$ since s is odd. Then

$$(g_1 s + x) + 2d' \equiv (g_1 s + x) + ((g_3 s + z) - (g_1 s + x)) = g_3 s + z \equiv z \pmod{s}$$
$$(g_1 s + x) + d' \equiv x + d \equiv y \pmod{s}.$$

With $\ell := (g_1 s + x) + d'$, the 3-AP $\{(g_1 s + x), \ell, (g_1 s + x) + 2d'\}$ is rainbow, because $\ell \in R_y$ and $P_y \subseteq P_0$, so $c(\ell) \neq \hat{c}(x) = c(g_1 s + x)$ and $c(\ell) \neq \hat{c}(z) = c(g_3 s + z)$.

In all cases, c has a rainbow 3-AP, contradicting our assumption (3). ☐

Next we prove two technical propositions used in the proof of Proposition 3.11, Propositions 3.9 and 3.10.

Proposition 3.9. *Let m and s be positive integers with s odd. Suppose c is a coloring of $\mathbb{Z}_{2^m s}$ using at least $r := \mathrm{aw}(\mathbb{Z}_s, 3) + 1$ colors that does not have a rainbow 3-AP. Let $R_0, R_1, \ldots, R_{s-1}$ be the residue classes modulo s in $\mathbb{Z}_{2^m s}$ with associated residue palettes P_i. Then $1 \leq |P_i| \leq 2$ for $i = 0, \ldots, s-1$, and all palettes P_i of size two share a common color.*

Proof. Since P_i is nonempty, $1 \leq |P_i|$. Observe that the coloring c of R_i induces a coloring on $\mathbb{Z}_{2^m}$ that uses only the colors in P_i and cannot contain a rainbow 3-AP. Thus $|P_i| \leq 2$ by Theorem 3.2, establishing the first statement.

By Proposition 3.7, each pair of residue palettes of size two must intersect. Suppose the palettes of size two do not all intersect in a common color. Then there are exactly three colors α, β, γ that are used by all the palettes of size two, and there are exactly three distinct palettes of size two, each consisting of two of these three colors. We show this configuration leads to a contradiction.

Create a coloring $\hat{c}$ of $\mathbb{Z}_s$ by the following method:

$$\hat{c}(i) = \begin{cases} c(i) & \text{if } |P_i| = 1, \\ \beta & \text{if } P_i = \{\alpha, \beta\}, \\ \text{the unique element of } P_i \setminus \{\gamma\} & \text{if } |P_i| = 2 \text{ and } \gamma \in P_i. \end{cases}$$

Observe that $\hat{c}$ uses r colors if there exists i such that $P_i = \{\gamma\}$ and $r - 1 = \mathrm{aw}(\mathbb{Z}_s, 3)$ colors otherwise, so in either case $\hat{c}$ must have a rainbow 3-AP. Suppose that $\{x, y, z\}$ is a rainbow 3-AP for the coloring $\hat{c}$ of $\mathbb{Z}_s$. Since $\hat{c}(x)$, $\hat{c}(y)$, and $\hat{c}(z)$ are distinct colors, at least one of the palettes P_x, P_y, P_z contains only one color. Consider the sizes of P_x, P_y, and P_z.

Case 1: $|P_z| = 1$. Observe that $\hat{c}(i)$ is always an element in P_i by our definition of $\hat{c}(i)$. Pick $n_1 \in R_x$ and $n_2 \in R_y$ such that $c(n_1) = \hat{c}(x)$ and $c(n_2) = \hat{c}(y)$. Thus $n_3 := 2n_2 - n_1$ is an element in R_z and so $c(n_3) = \hat{c}(z)$. Since $\hat{c}(x), \hat{c}(y), \hat{c}(z)$ are all distinct, $\{n_1, n_2, n_3\}$ is a rainbow 3-AP. The case $|P_x| = 1$ is symmetric.

Case 2: $|P_x| = |P_z| = 2$ and $|P_y| = 1$. Since $\hat{c}(x) \neq \hat{c}(z)$, it must be that $\{\hat{c}(x), \hat{c}(z)\} = \{\alpha, \beta\}$. Without loss of generality, we assume that $\hat{c}(x) = \beta$ and $\hat{c}(z) = \alpha$. By the definition of $\hat{c}$, $P_z = \{\alpha, \gamma\}$. Then P_x is one of $\{\alpha, \beta\}$ or $\{\beta, \gamma\}$. If $\hat{c}(y) \notin P_x \cup P_z$, then any 3-AP $\{n_1, n_2, n_3\}$ where $n_1 \in R_x$ and $c(n_1) = \beta$, $n_2 \in R_y$, and $n_3 \in R_z$ is a rainbow 3-AP in the original coloring. Thus, $\hat{c}(y) \in P_x \cup P_z \subseteq \{\alpha, \beta, \gamma\}$, but $\hat{c}(y) \notin \{\alpha, \beta\} = \{\hat{c}(x), \hat{c}(z)\}$, so $\hat{c}(y) = \gamma$. Note that this implies $\hat{c}$ uses all r colors.

Since this is the final case, and all previous cases led to contradictions, every rainbow 3-AP in $\mathbb{Z}_s$ given by the coloring $\hat{c}$ must be of the form $\{x, y, z\}$ where $\{\hat{c}(x), \hat{c}(z)\} = \{\alpha, \beta\}$ and $\hat{c}(y) = \gamma$. Create a new coloring c' of $\mathbb{Z}_s$ where $c'(i) = \begin{cases} c(i) & \text{if } \hat{c}(i) \neq \gamma, \\ \beta & \text{if } \hat{c}(i) = \gamma. \end{cases}$

Now, every 3-AP that was previously non-rainbow in $\hat{c}$ remains non-rainbow in c' and the rainbow 3-APs (which necessarily used the colors α, β, and γ) are no longer rainbow. Thus, this coloring c' does not have a rainbow 3-AP, but c' uses $r - 1 = \mathrm{aw}(\mathbb{Z}_s, 3)$ colors, a contradiction.

The above cases show that having no common color among the palettes of size two leads to a contradiction. Therefore, all of the residue palettes of size two share a common color. ☐

Proposition 3.10. *Suppose c is a coloring of $\mathbb{Z}_{2t}$ ($t \geq 1$) that does not have a rainbow 3-AP. Let A and B denote the residue palettes modulo 2 in $\mathbb{Z}_{2t}$ associated with the even and odd numbers, respectively. Then $|A \setminus B| \leq 1$ and $|B \setminus A| \leq 1$.*

Proof. It suffices to show that $|A \setminus B| \leq 1$ for every such coloring c because if $|B \setminus A| \geq 2$, then the coloring defined by the rotation $c'(x) := c(x+1)$ has the roles of A and B reversed. Suppose not, so there exist two colors α, γ that appear only in A. Let $n_1 = 2m_1$ and $n_3 = 2m_3$ be even elements such that $c(n_1) = \alpha$ and $c(n_3) = \gamma$. We can select m_1 and m_3 such that $0 \leq m_1 < m_3 < t$. Performing arithmetic in the integers, we can choose $m_3 - m_1$ to be minimum with respect to the fact that the set of colors $\{c(2m_1), c(2m_3)\}$ is $\{\alpha, \gamma\}$. Let $n_2 = m_1 + m_3$ and observe that $\{n_1, n_2, n_3\}$ is a 3-AP and hence is not rainbow. Therefore, n_2 must have the color α or γ and thus is even. However, this implies that $n_2 = 2m_2$ and $m_1 < m_2 < m_3$, while one of the sets of colors $\{c(2m_1), c(2m_2)\}$ or $\{c(2m_2), c(2m_3)\}$ is $\{\alpha, \gamma\}$, so one of the pairs (m_1, m_2), (m_2, m_3) violates our extremal choice. □

Proposition 3.11. *Let m and s be positive integers with s odd. Then*

$$\text{aw}(\mathbb{Z}_{2^m s}, 3) \leq \text{aw}(\mathbb{Z}_s, 3) + 1.$$

Proof. The result is immediate for $s = 1$ because $\text{aw}(S, 3) = |S| + 1$ for $|S| < 3$, so assume $s \geq 3$. We proceed by induction on m. Suppose c is an exact r-coloring of $\mathbb{Z}_{2^m s}$ with $r = \text{aw}(\mathbb{Z}_s, 3) + 1$ that does not have a rainbow 3-AP. Let A and B denote the residue palettes of the even and odd numbers, respectively. By Proposition 3.10, $|A \setminus B| \leq 1$ and $|B \setminus A| \leq 1$, so $|B| \geq r - 1$ and $|A| \geq r - 1$. The base case $m = 1$ is then immediate, because the coloring of the even numbers of $\mathbb{Z}_{2s}$ induces a coloring of $\mathbb{Z}_s$, so a rainbow 3-AP necessarily exists, producing a contradiction.

Now consider $m > 1$. As usual $R_i, i = 0, \dots, s-1$, are the residue classes modulo s of $\mathbb{Z}_{2^m s}$ and $P_i, i = 0, \dots, s-1$, are the residue palettes. For $0 \leq i \leq s-1$, let $A_i = P_i \cap A$ be the colors appearing on the even numbers in R_i, and let $B_i = P_i \cap B$ be the colors appearing on the odd numbers in R_i. Thus, $P_i = A_i \cup B_i$, $A = \bigcup_{i=0}^{s-1} A_i$, and $B = \bigcup_{i=0}^{s-1} B_i$. We claim that $|A| = |B| = r - 1$. To see this, observe that the even elements induce a coloring of $\mathbb{Z}_{2^{m-1}s}$, so if $|A| = r$, then a rainbow 3-AP necessarily exists, since $r \geq \text{aw}(\mathbb{Z}_{2^{m-1}s}, 3)$ by the induction hypothesis. Thus $|A| \leq r - 1$, and so $|A| = r - 1$. The proof that $|B| = r - 1$ is analogous.

Since $|A| = |B| = r - 1$, there exist colors α, β such that $A \setminus B = \{\alpha\}$ and $B \setminus A = \{\beta\}$. Assume $\alpha \in A_u$ and $\beta \in B_v$. Let $j = v - u$, hence $\beta \in B_{u+j} = B_v$. Since there is no rainbow 3-AP, $u + 2j$ must have a color in palette A, $\alpha \in A_{u+2j}$, which then implies $\beta \in B_{u+3j} = B_{v+2j}$. Iterating this process gives that $\alpha \in A_{u+2\ell j}$ and $\beta \in B_{v+2\ell j}$ for all $\ell \geq 0$. Since s is odd, we have that for all $q \geq 0$, A_{u+qj} is of the form $A_{u+2\ell j}$ for some ℓ and similarly, every $B_{u+qj} = B_{v+(q-1)j}$ is of the form $B_{v+2\ell j}$ for some ℓ. Therefore, $P_{u+qj} = \{\alpha, \beta\}$ for all $q \geq 0$. By Proposition 3.9, there is a common color for palettes of size two, and thus one of α or β is this common color. Without loss of generality, assume that α is the common color for all palettes. This implies that $|B_i| = 1$ for all $0 \leq i \leq s-1$. Hence, defining $\hat{c}(i)$ to be the unique color in B_i defines an exact $(r-1)$-coloring of $\mathbb{Z}_s$ that avoids rainbow 3-APs. However, $r - 1 = \text{aw}(\mathbb{Z}_s, 3)$, a contradiction. □

Proposition 3.5 is now established from Proposition 3.8 and Proposition 3.11.

Definition 3.12. Let $n \geq 3$ be a a fixed integer. Define f_2 to be 0 if n is odd and 1 if n is even. For $a \geq 3$, define f_a to be the number of odd prime factors p (counted according to multiplicity) having $\text{aw}(\mathbb{Z}_p, 3) = a$. Define mv to be the maximum value of $\text{aw}(\mathbb{Z}_p, 3)$ over all odd prime factors p of n (or 0 if n has no odd prime factors).

Lemma 3.13 below follows from Lemma 3.2, Propositions 3.4 and Proposition 3.5 by induction.

Lemma 3.13. *For any integer $n \geq 3$,*

$$2 + f_2 + \sum_{a=3}^{\text{mv}} f_a \leq \text{aw}(\mathbb{Z}_n, 3) \leq 2 + f_2 + \sum_{a=3}^{\text{mv}} (a-2) f_a.$$

If $\mathrm{aw}(\mathbb{Z}_{pt}, 3) \geq \mathrm{aw}(\mathbb{Z}_t, 3) + \mathrm{aw}(\mathbb{Z}_p, 3) - 2$ for every odd prime p and $t \geq 3$ (see Conjecture 5.5), then for any n, $\mathrm{aw}(\mathbb{Z}_n, 3)$ can be computed from the values of $\mathrm{aw}(\mathbb{Z}_p, 3)$ for primes p, because the second inequality in Lemma 3.13 becomes an equality. Proposition 3.14 and Proposition 3.16 below establish this property for certain odd numbers using a special type of coloring, allowing computation of $\mathrm{aw}(\mathbb{Z}_n, 3)$ from values for primes whenever all prime factors of n are less than 100 (see Corollary 3.15 below). A coloring c of $\mathbb{Z}_n$ is a *singleton extremal coloring* if c is an exact $(\mathrm{aw}(\mathbb{Z}_n, 3) - 1)$-coloring of $\mathbb{Z}_n$ with no rainbow 3-AP such that some color is used exactly once.

Proposition 3.14. *Suppose s is odd and $\mathbb{Z}_s$ has a singleton extremal coloring. Then for $t \geq 3$,*

$$\mathrm{aw}(\mathbb{Z}_{st}, 3) \geq \mathrm{aw}(\mathbb{Z}_t, 3) + \mathrm{aw}(\mathbb{Z}_s, 3) - 2.$$

Proof. Let c_s be a singleton extremal coloring of $\mathbb{Z}_s$. Note that we can shift c_s so that $c_s(0)$ is the color that is used exactly once. Choose a coloring c_t of $\mathbb{Z}_t$ using $\mathrm{aw}(\mathbb{Z}_t, 3) - 1$ colors not used by c_s that does not have a rainbow 3-AP. Let $R_0, R_1, \ldots, R_{s-1}$ be the residue classes modulo s in $\mathbb{Z}_{st}$. Define a coloring $\hat{c}$ of $\mathbb{Z}_{st}$ as follows: For $i = 1, \ldots, s-1$ and $\ell \in R_i$, $\hat{c}(\ell) := c_s(i)$, and for $0 \leq j \leq t-1$, $\hat{c}(js) := c_t(j)$. Notice that we now have an exact $\mathrm{aw}(\mathbb{Z}_s, 3) - 2 + \mathrm{aw}(\mathbb{Z}_t, 3) - 1$ coloring of $\mathbb{Z}_{st}$ because we have removed color $c_s(0)$. Clearly, if a 3-AP is within some residue class it is not rainbow. Because s is odd, $d \not\equiv 0 \pmod{s}$ implies $2d \not\equiv 0 \pmod{s}$ and $2d \not\equiv d \pmod{s}$, so a 3-AP that is not entirely within one residue class has elements in three different residue classes. But a rainbow 3-AP with elements in three different residue classes would imply a rainbow 3-AP in c_s, which does not exist. So we have found a coloring of $\mathbb{Z}_{st}$ using $\mathrm{aw}(\mathbb{Z}_t, 3) + \mathrm{aw}(\mathbb{Z}_s, 3) - 3$ colors that does not have a rainbow 3-AP. Thus $\mathrm{aw}(\mathbb{Z}_{st}, 3) \geq \mathrm{aw}(\mathbb{Z}_t, 3) + \mathrm{aw}(\mathbb{Z}_s, 3) - 2$. □

Corollary 3.15. *If for every odd prime factor p of $n \geq 3$, $\mathrm{aw}(\mathbb{Z}_p, 3) = 3$ or $\mathbb{Z}_p$ has a singleton extremal coloring, then we can determine $\mathrm{aw}(\mathbb{Z}_n, 3)$ from the values of $\mathrm{aw}(\mathbb{Z}_p, 3)$ for the prime factors p:*

$$\mathrm{aw}(\mathbb{Z}_n, 3) = 2 + f_2 + \sum_{a=3}^{\mathrm{mv}(n)} (a-2) f_a. \tag{4}$$

Although the conditions on Corollary 3.15 seem restrictive, it happens that all of the values of $\mathrm{aw}(\mathbb{Z}_p, 3)$ we can compute, for p a prime, result from a singleton coloring of $\mathbb{Z}_p$ as we see in Proposition 3.16. In Question 5.6, we ask if this is the case for all primes.

Proposition 3.16. *For all primes $p < 100$, $\mathrm{aw}(\mathbb{Z}_p, 3) = 3$ if $p \notin Q_4 := \{17, 31, 41, 43, 73, 89, 97\}$ and $\mathrm{aw}(\mathbb{Z}_p, 3) = 4$ if $p \in Q_4$. Furthermore, for every prime $p < 100$, $\mathbb{Z}_p$ has a singleton extremal coloring.*

p	Singleton extremal coloring of $01 \cdots (p-1)$
17	31121222112221211
31	3112122212222221122222212221211
41	31121122111222221212112121222221112211211
43	3111111212111222122111111221222111212111111
73	3112122211222222121222222222222212221122212222222222222121222222112221211
89	31111111111111212111211111122122211121221111111111122121112221221111112111212 1111111111111
97	3111121211211222121222121121222111211222222112211112211222222112111222121121222121222112112121111

Table 3: Singleton extremal coloring of $\mathbb{Z}_p$ for primes $p < 100$ with $\mathrm{aw}(\mathbb{Z}_p, 3) = 4$.

Proof. The statement that for any prime $p < 100$, $\mathrm{aw}(\mathbb{Z}_p, 3) = 3$ if $p \notin Q_4$ and $\mathrm{aw}(\mathbb{Z}_n, 3) = 4$ if $p \in Q_4$ has been verified computationally (see Table 2). If $\mathrm{aw}(\mathbb{Z}_p, 3) = 3$, then $c(0) = 1, c(i) = 2$ for $i > 0$ is a singleton extremal coloring of $\mathbb{Z}_p$. For each $p \in Q_4$, a singleton extremal coloring of $\mathbb{Z}_p$ is given in Table 3. □

If every prime factor p of n is less than 100, then (4) applies and can be simplified to Equation (1) in Theorem 1.5.

Example 3.17. Let $n = 14,582,937,583,067,568$. Since $n = 2^4 \cdot 3 \cdot 11^2 \cdot 13 \cdot 17^2 \cdot 53^3 \cdot 67^2$, $\text{aw}(\mathbb{Z}_n, 3) = 3 + 9 + 2 \cdot 2 = 16$ by applying (1) (see Table 2 for the values of $\text{aw}(\mathbb{Z}_p, 3)$).

Remark 3.18. If for every prime factor p of n, $\mathbb{Z}_p$ has a singleton extremal coloring, then the constructive proof of Lemma 3.14 gives a singleton extremal coloring of $\mathbb{Z}_n$.

3.2 Main results for $\text{aw}(\mathbb{Z}_n, k), k \geq 4$

In this section, we specialize to the case where $k \geq 4$ and prove Theorem 1.6. Corollary3.19 below, which follows from Corollary 2.14 and Remark 1.4, gives us $ne^{-\log\log\log n - \omega(1)}$ as an upper bound for $\text{aw}(\mathbb{Z}_n, k)$.

Corollary 3.19. *For every fixed positive integer k, $\text{aw}(\mathbb{Z}_n, k) = o\left(\frac{n}{\log\log n}\right)$.*

Our lower bound for $\text{aw}(\mathbb{Z}_n, k)$ when $n > 12$ is presented in Lemma 3.20.

Lemma 3.20. *There exists an absolute constant $b > 0$ such that for all $c > 3$, $\frac{n}{c} \geq 4$ and $k \geq 4$,*

$$\text{aw}(\mathbb{Z}_n, k) > \left(\frac{n}{c}\right) e^{-b\sqrt{\log(n/c)}} = ne^{-b\sqrt{\log(n/c)} - \log c} = n^{1-o(1)}.$$

Lemma 3.20 is proven using the Behrend construction from Section 2.2 and Proposition 3.21 below. The Behrend construction in the integers $\{1, \ldots, m\}$ has no punctured 4-AP and size $me^{-b\sqrt{\log m}}$ for some absolute constant b.

Proposition 3.21. *Let $c > 3$ be a real number, and let $\left[\frac{n}{c}\right]$ denote the first $\lfloor \frac{n}{c} \rfloor$ consecutive residues in $\mathbb{Z}_n$. Suppose $S \subseteq \left[\frac{n}{c}\right]$ does not contain any punctured 4-APs. Then $\text{aw}(\mathbb{Z}_n, k) > |S| + 1$ for all $k \geq 4$.*

Proof. Color each member of S a distinct color, and color each member of $\mathbb{Z}_n \setminus S$ with a new color called *zero*. Note that because $c > 3$, each $i \in \mathbb{Z}_n$ with $\frac{n}{c} \leq i < n$ will be colored *zero*. If $K = \{a_1, a_2, a_3, a_4\}$ is a rainbow 4-AP in $\mathbb{Z}_n$, then at most one element of K is not in S. Without a loss of generality, assume $a_3, a_4 \in S$. Then there exists $d \in \mathbb{Z}$ such that $d \equiv a_4 - a_3 \pmod{n}$ and $|d| \leq \frac{n}{c}$.

Suppose $a_2 \in S$. Because $|d| \leq \frac{n}{c} < \frac{n}{2}$, we must have that a_2, a_3, a_4 is a 3-AP in $\left[\frac{n}{c}\right]$. This contradicts the fact that S contains no punctured 4-APs, so we must have $a_2 \notin S$ and $a_1 \in S$. However, since $2|d| \leq \frac{2n}{c} < \frac{(c-1)n}{c}$, we must have that a_1, a_3, a_4 is a punctured 4-AP in $\left[\frac{n}{c}\right]$. This is a contradiction, so $a_1 \notin S$.

This means that K could not have been rainbow, so we have a $(|S| + 1)$-coloring of $\mathbb{Z}_n$ with no rainbow 4-APs. □

Altogether, our bounds for $\text{aw}(\mathbb{Z}_n, k)$, $k \geq 4$ are

$$ne^{-b\sqrt{\log(n/c)} - \log c} < \text{aw}(\mathbb{Z}_n, k) \leq ne^{-\log\log\log n - \omega(1)}.$$

This completes the proof of Theorem 1.6.

3.3 Additional results for $\text{aw}(\mathbb{Z}_n, k)$

In this section, we present computed data for $\text{aw}(\mathbb{Z}_n, k), k \geq 4$, establish the value of $\text{aw}(\mathbb{Z}_n, k)$ for $k = n$, $n - 1$, and $n - 2$, and present some examples that show some additional results fail to extend from $[n]$ to $\mathbb{Z}_n$. Table 4 below lists the computed values of $\text{aw}(\mathbb{Z}_n, k)$ for $k = 4, \ldots, n$ in the row labeled n.

Next we examine $\text{aw}(\mathbb{Z}_n, k)$ for k close to n.

Proposition 3.22. *For positive n and k we have $\text{aw}(\mathbb{Z}_n, k) = n$ if and only if $k = n$.*

Proof. If $k = n$ the result is obvious. Now suppose that $k < n$ and consider an exact $(n-1)$-coloring of $\mathbb{Z}_n$. Then there are two numbers with the same color and all other numbers are colored distinctly. Suppose x and y are the the two numbers with the same color. Then $\{x + 1, ..., x + k\}$ is a k-AP that does not contain x, and so is rainbow. Therefore $\text{aw}(\mathbb{Z}_n, k) \leq n - 1$. □

$n \setminus k$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	4															
5	4	5														
6	5	5	6													
7	4	5	6	7												
8	6	6	7	7	8											
9	5	6	8	8	8	9										
10	6	8	8	8	9	9	10									
11	5	6	7	8	9	9	10	11								
12	8	9	10	10	11	11	11	11	12							
13	5	7	8	9	10	10	11	11	12	13						
14	6	8	10	12	12	12	12	12	13	13	14					
15	8	11	12	12	12	13	14	14	14	14	14	15				
16	8	10	10	11	14	14	14	14	15	15	15	15	16			
17	6	8	10	11	12	12	13	14	14	15	15	15	16	17		
18	8	10	13	14	14	16	16	16	17	17	17	17	17	17	18	
19	6	9	10	12	12	14	14	15	16	16	16	17	17	17	18	19

Table 4: Computed Values of $\mathrm{aw}(\mathbb{Z}_n, k)$ for $k \geq 4$.

Corollary 3.23. *For positive n,* $\mathrm{aw}(\mathbb{Z}_n, n-1) = n-1$.

A pattern can be observed in the values of $\mathrm{aw}(\mathbb{Z}_n, n-2)$, and this is established in Proposition 3.24.

Proposition 3.24. *For positive $n \geq 5$, if n is prime then* $\mathrm{aw}(\mathbb{Z}_n, n-2) = n-2$*; otherwise* $\mathrm{aw}(\mathbb{Z}_n, n-2) = n-1$.

Proof. We trivially have a lower bound of $n-2$ for $\mathrm{aw}(\mathbb{Z}_n, n-2)$. First we assume n is prime. We claim that for any two distinct elements x and y there is an $(n-2)$-AP that misses x and y. To see this, simply form the n-AP with $a = x$ and $d = (y-x)$, this will cover all of $\mathbb{Z}_n$ and now removing the first two terms leaves us with an $(n-2)$-AP that does not contain x or y. So suppose we have an exact $(n-2)$-coloring. Then either there is one color that occurs three times or two colors that each occur twice, and in either case all other colors occur exactly once. In either case we can choose two numbers to avoid and then the remaining $n-2$ numbers are rainbow, but as just noted above the remaining $n-2$ numbers are an arithmetic progression. Therefore every $(n-2)$-coloring contains a rainbow progression.

When n is not a prime, let p be the smallest prime divisor of n and consider the $(n-2)$-coloring formed by coloring 0, p and $2p$ monochromatically, with the remaining numbers all given distinct colors. This is an $(n-2)$-coloring (since $2p < n$ by assumption that $n \geq 5$). We claim this coloring has no rainbow $(n-2)$-AP (along with the upper bound of $n-1$, this claim establishes the result). Suppose that $K = \{a, a+d, ..., a+(n-3)d\}$ is a rainbow $(n-2)$-AP, so all the elements of K are distinct and K necessarily misses two of $0, p, 2p$. Since $\mathbb{Z}_n$ cannot have a proper subgroup of order $n-2$, extending K to a n-AP necessarily produces all elements of $\mathbb{Z}_n$ and thus $\{a+(n-2)d, a+(n-1)d\} \subseteq \{0, p, 2p\}$. But then we have that p divides $d = ((a+(n-1)d) - (a+(n-2)d))$, showing that this arithmetic progression can have at most $\frac{n}{p} < n-2$ terms, which is a contradiction. ☐

Proposition 3.22 shows that the "if" direction of Theorem 2.16 ($k \geq \lceil \frac{n}{2} \rceil + 1$ implies $\mathrm{aw}([n], k) = n$) does not extend to $\mathbb{Z}_n$. Example 3.25 below shows that the extension of Proposition 2.15 to $\mathbb{Z}_n$, which would assert that $\mathrm{aw}(\mathbb{Z}_n, k) \leq \mathrm{aw}(\mathbb{Z}_{n-1}, k) + 1$, is not true in general. There are counterexamples in both the cases $k = 3$ and $k \geq 4$.

Example 3.25. According to our computed data (see Table 2 in Section 3.1), $\mathrm{aw}(\mathbb{Z}_{30}, 3) = 5$ and $\mathrm{aw}(\mathbb{Z}_{29}, 3) = 3$. Furthermore, $\mathrm{aw}(\mathbb{Z}_8, 4) = 6$ and $\mathrm{aw}(\mathbb{Z}_7, 4) = 4$ (see Table 4 in Section 3.2).

Example 3.26 below shows that Theorem 2.17, which bounds the anti-van der Waerden number of a sum in terms of the anti-van der Waerden numbers of the summands, does not extend to $\mathbb{Z}_n$.

Example 3.26. According to our computed data (see Table 4 in Section 3.2),

$$\mathrm{aw}(\mathbb{Z}_{12}, 4) = 8 > 4 + 4 - 1 = \mathrm{aw}(\mathbb{Z}_5, 4) + \mathrm{aw}(\mathbb{Z}_7, 4) - 1.$$

There are also examples for $k = 3$, such as $\mathrm{aw}(\mathbb{Z}_{54}, 3) = 6 > 3 + 3 - 1 = \mathrm{aw}(\mathbb{Z}_{47}, 3) + \mathrm{aw}(\mathbb{Z}_7, 3) - 1$.

4 Computation

Many of the results we have proved in this paper were first conjectured from examination of data. In this section, we briefly discuss an efficient algorithm to find an exact r-coloring of $[n]$ or $\mathbb{Z}_n$ that avoids a rainbow k-AP, if such a coloring exists. For the sake of brevity, we will focus on the case of coloring $[n]$ since this case has a few extra properties that the $\mathbb{Z}_n$ case does not. Specifically, we have $[m] \subseteq [n]$ for all $m \leq n$ while $\mathbb{Z}_n$ contains a copy of $\mathbb{Z}_m$ if and only if m divides n.

Fix k, n, and r and assume that all values of $\mathrm{aw}([m], k)$ have been computed for $k \leq m < n$. Let $c : [n] \to [r] \cup \{*\}$ be a function called a *partial r-coloring*, where every position i has color $c(i) \in [r]$ or $c(i) = *$ and i is *uncolored*. By starting with all positions uncolored, we recursively attempt to extend a partial r-coloring c where the positions in $[i]$ are colored to an exact r-coloring c' that avoids rainbow k-APs. We branch at each recursive call for all possible choices of color for $c(i+1)$ such that no k-AP within $[i+1]$ is colored with k distinct colors. To guarantee that no chosen color creates a rainbow k-AP, we maintain a list of sets $D(j) \subseteq [r]$ that contain all of the possible colors for the position j. Specifically, assigning $c(j)$ to be any color in $[r] \setminus D(j)$ will immediately create a rainbow k-AP. Whenever a color is assigned to a position i, we consider a k-AP, X, whose second-to-last element is i. If the set $c(X) = \{c(i') : i' \in X - \max X\}$ contains $k - 1$ distinct colors, we say that X is an *almost-rainbow k-AP* and the color for $\max X$ must be one of these $k - 1$ colors. Therefore, we can update $D(\max X)$ to be $D(\max X) \cap c(X)$. For simplicity, we update $D(i)$ to be $\{c(i)\}$ when i is assigned the color $c(i)$.

We can also make a few small adjustments to greatly reduce the search space. First, we assume that the coloring c is lexicographically-minimum: for two colors $a, b \in [r]$ with $a < b$, we assume that the first position with color a appears before the first position with color b. Second, the domains $D(j)$ contain the possible colors for the positions that remain uncolored. If $\bigcup_{j\in[n]} D(j) \neq [r]$, then c cannot extend to an exact r-coloring. Finally, if the first i positions are all colored with the color 1, then for any extension of c to an exact r-coloring of $[n]$, the last $n - i + 2$ positions form an exact r-coloring. Thus, if $\mathrm{aw}([n-i+1], k) \leq r$, then it is impossible to extend c to an exact r-coloring of $[n]$ without creating a rainbow k-AP.

Our recursive algorithm is given as Algorithm 1 and is initialized by Algorithm 2. Similar algorithms are implemented for the case of r-coloring $\mathbb{Z}_n$. All source code and data are available online[1] including computed values of $\mathrm{aw}([n], k)$ and $\mathrm{aw}(\mathbb{Z}_n, k)$, extremal colorings, and reports of computation time.

5 Conjectures and open questions

We conclude by summarizing some open questions and conjectures.

Uherka [13] observed that $\mathrm{aw}([n], 3)$ is not a monotone function in n, as there are values of n where $\mathrm{aw}([n], 3) = \mathrm{aw}([n-1], 3) - 1$. Does this happen infinitely often? Are larger drops possible?

Conjecture 5.1. *For positive integers n and k,* $\mathrm{aw}([n], k) \geq \mathrm{aw}([n-1], k) - 1$.

Conjecture 1.2 states that the lower bound $\mathrm{aw}([n], 3) \geq \lceil \log_3 n \rceil + 2$ is correct to within an additive constant. We further conjecture that the lower bound in Lemma 2.3 is in fact the exact value when n is a power of three. It is true for the computed data available (see Remark 2.1).

Conjecture 5.2. *Let m be a nonnegative integer. Then* $\mathrm{aw}([3^m], 3) = m + 2$.

Question 5.3. *Is it true that* $\mathrm{aw}([3n], 3) = \mathrm{aw}([n], 3) + 1$ *for all positive integers n?*

[1]All source code and data can be found at `http://www.math.iastate.edu/dstolee/r/rainbowaps.htm`

Algorithm 1 FindColorings$(k, r, n, \text{aw}, c, D, i)$ – Find exact D-colorings on $[n]$ that avoid rainbow k-APs and extend the coloring c on $[i-1]$. Assume $\text{aw}([m], k)$ is known for all $m < n$.

```
if i ≡ n then
  output c
  return
else if ∪_{j∈[n]} D(j) ≠ [r] then
  return // This coloring cannot extend to an exact r-coloring!
else if i > 2 and ∀j < i, c(j) ≡ 1 and aw([n − i + 2], k) ≤ r then
  return // An exact r-coloring extending c induces an exact r-coloring on {i − 1, ..., n}.
end if
M ← max{c(j) : j < i} ∪ {0}
// Attempt all colors in the domain D(i) that are at most M + 1.
for all a ∈ D(i) ∩ [M + 1] do
  c(i) ← a,    D(i) ← {a}
  // Update all domains D′(t) when almost-rainbow k-APs exist.
  D′ ← D
  for all d ∈ {1, ..., ⌊i/(k − 2)⌋} do
    A ← ∅
    for all ℓ ∈ {0, ..., k − 2} do
      t ← i − ℓ · d
      A ← A ∪ {c(t)}
    end for
    if |A| ≡ k − 1 then
      t ← i + d
      D′(t) ← D′(t) ∩ A
    end if
  end for
  call FindColorings(k, r, n, aw, c, D′, i + 1)
end for
```

Algorithm 2 FindColoring(k, r, n, aw) – Find exact r-colorings on $[n]$ that avoid rainbow k-APs.

```
for all i ∈ [n] do
  c(i) ← ∗
  D(i) ← [r]
end for
call FindColorings(k, r, n, aw, c, D, 1)
```

Recall that a singleton extremal coloring of S is an exact coloring of S that avoids rainbow k-APs and uses exactly $\text{aw}(S, k) - 1$ colors. Singleton extremal colorings exist for all of our computed examples where $S = \mathbb{Z}_n$ and $k = 3$, as we have seen by the computer computations described in Section 4 for every value of $n \leq 58$. Is this always the case?

Conjecture 5.4. *For $k = 3$, there exists a singleton extremal coloring of $[n]$ and of $\mathbb{Z}_n$.*

Proposition 3.14 gives an effective lower bound on $\text{aw}([n], 3)$ for n composite with certain odd factors. If Conjecture 5.4 holds for $\mathbb{Z}_n$, then the condition for Proposition 3.14 applies to all factors, implying the following conjecture.

Conjecture 5.5. *For p an odd prime and $t \geq 3$, $\text{aw}(\mathbb{Z}_{pt}, 3) \geq \text{aw}(\mathbb{Z}_t, 3) + \text{aw}(\mathbb{Z}_p, 3) - 2$.*

If this conjecture holds, then computing $\text{aw}(\mathbb{Z}_n, 3)$ depends only on the values of $\text{aw}(\mathbb{Z}_p, 3)$ for the prime factors p of n. Since $3 \leq \text{aw}(\mathbb{Z}_p, 3) \leq 4$ for all primes $p < 100$, we consider which primes allow $\text{aw}(Z_p, 3) = 3$.

Question 5.6. *Are there infinitely many primes p such that $\text{aw}(\mathbb{Z}_p, 3) = 3$?*

Question 5.7. *Is $\text{aw}(\mathbb{Z}_p, 3) > 3$ for all p prime, $p \equiv 1 \pmod 8$?*

Question 5.8. *Does there exist a prime p such that $\text{aw}(\mathbb{Z}_p, 3) \geq 5$?*

One suggested approach to finding primes p for which $\text{aw}(\mathbb{Z}_p, 3) = 3$ is to search for primes p such that the multiplicative group $\mathbb{Z}_p^\times$ is generated by 2. In that case, a singleton extremal coloring of $\mathbb{Z}_p$ would have only two colors. However, the existence of an infinite family of such primes is still open.

Conjecture 5.9 (Artin's Conjecture)**.** [11, p. 217] *There are infinitely many primes p such that 2 is a generator of the multiplicative group $\mathbb{Z}_p^\times$.*

If Artin's Conjecture holds, we suspect it would give us an infinite family of $\mathbb{Z}_p$ such that $\text{aw}(\mathbb{Z}_p, 3) = 3$.

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