Color-blind index in graphs of very low degree*

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Abstract

Let $c : E(G) \to [k]$ be an edge-coloring of a graph G, not necessarily proper. For each vertex v, let $\bar{c}(v) = (a_1, \ldots, a_k)$, where a_i is the number of edges incident to v with color i. Reorder $\bar{c}(v)$ for every v in G in nonincreasing order to obtain $c^*(v)$, the color-blind partition of v. When c^* induces a proper vertex coloring, that is, $c^*(u) \neq c^*(v)$ for every edge uv in G, we say that c is color-blind distinguishing. The minimum k for which there exists a color-blind distinguishing edge coloring $c : E(G) \to [k]$ is the color-blind index of G, denoted $\operatorname{dal}(G)$. We demonstrate that determining the color-blind index is more subtle than previously thought. In particular, determining if $\operatorname{dal}(G) \leq 2$ is NP-complete. We also connect the color-blind index of a regular bipartite graph to 2-colorable regular hypergraphs and characterize when $\operatorname{dal}(G)$ is finite for a class of 3-regular graphs.

1 Introduction

Coloring the vertices or edges of a graph G in order to distinguish neighboring objects is fundamental to graph theory. While typical coloring problems color the same objects that they aim to distinguish, it is natural to consider how edge-colorings can distinguish neighboring vertices. For an edge-coloring c using colors $\{1, \ldots, k\}$, the *color partition* of a vertex v is given as $\bar{c}(v) = (a_1, \ldots, a_k)$, where the integer a_i is the number of edges incident to v with color i. The edge-coloring c is *neighbor distinguishing* if $\bar{c}$ is a proper vertex coloring of the vertices of G. The *neighbor-distinguishing index* of G is the minimum k such that there exists a neighbor distinguishing k-edge-coloring of G. Define $c^*(v)$ to be the list $\bar{c}(v)$ in nonincreasing order; call $c^*(v)$ the *color-blind partition* at v, since $c^*(v)$ allows for counting the sizes of the color classes incident to v without identifying the colors. The edge-coloring c is *color-blind distinguishing* if c^* is a proper vertex coloring of the vertices of G. The *color-blind index* of G, denoted $\operatorname{dal}(G)$, is the minimum k such that there exists a color-blind distinguishing k-edge-coloring of G.

The neighbor-distinguishing index and color-blind index do not always exist for a given graph G. A graph G has no neighbor-distinguishing coloring if and only if it contains a component containing a single edge [4]. The conditions that guarantee G has a color-blind distinguishing coloring are unclear. When a graph G has no color-blind distinguishing coloring, we say that $\operatorname{dal}(G)$ is undefined or write $\operatorname{dal}(G) = \infty$. Kalinowski, Pilśniak, Przybyło, and Woźniak [9] defined color-blind distinguishing colorings and presented

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several examples of graphs that have no color-blind distinguishing colorings. All of the known examples that fail to have color-blind distinguishing colorings have minimum degree at most three.

When two adjacent vertices have different degree, their color-blind partitions are distinct for every edge-coloring. Thus, it appears that constructing a color-blind distinguishing coloring is most difficult when a graph is regular and of small degree. Most recent work [1, 11] has focused on demonstrating that $\mathrm{dal}(G)$ is finite and small when G is a regular graph (or is almost regular) of large degree. These results were improved by Przybyło [12] in the following theorem.

Theorem 1 (Przybyło [12])**.** *If G is a graph with minimum degree $\delta(G) \geq 3462$, then* $\mathrm{dal}(G) \leq 3$.

We instead focus on graphs with very low minimum degree. In Section 2, we demonstrate that it is difficult to determine $\mathrm{dal}(G)$, even when it is promised to exist.

Theorem 2. *Determining if* $\mathrm{dal}(G) = 2$ *is* NP*-complete, even under the promise that* $\mathrm{dal}(G) \in \{2, 3\}$.

The hardness of determining $\mathrm{dal}(G)$ implies that there is no efficient characterization of graphs with low color-blind index (assuming $\mathsf{P} \neq \mathsf{NP}$). Therefore, we investigate several families of graphs with low degree in order to determine their color-blind index. For example, it is not difficult to demonstrate that $\mathrm{dal}(G) \leq 2$ when G is a tree on at least three vertices.

A 2-regular graph is a disjoint union of cycles, and the color-blind index of cycles is known [9], so we pursue the next case by considering different classes of 3-regular graphs, and determine if they have finite or infinite color-blind index. If G is a k-regular bipartite graph, then the color-blind index of G is at most 3 [9]. In Section 3, we demonstrate that a k-regular bipartite graph has color-blind index 2 exactly when it is associated with a 2-colorable k-regular k-uniform hypergraph. Thomassen [13] and Henning and Yeo [8] proved that all k-regular, k-uniform hypergraphs are 2-colorable when $k \geq 4$; this demonstrates that all k-regular bipartite graphs have color-blind index at most 2 when $k \geq 4$. Thus, for k-regular bipartite graphs it is difficult to distinguish between color-blind index 2 or 3 only when $k = 3$.

To further investigate 3-regular graphs, we consider graphs that are very far from being bipartite in Section 4. In particular, we consider a connected 3-regular graph G where every vertex is contained in a 3-cycle. If there is a vertex in three 3-cycles, then G is isomorphic to K_4 and there does not exist a color-blind distinguishing coloring of G [9]. If v is a vertex in two 3-cycles, then one of the neighbors u of v is in both of those 3-cycles. These two 3-cycles form a *diamond*. We say G is a *cycle of diamonds* if G is a 3-regular graph where every vertex in G is in a diamond; G is an *odd cycle of diamonds* if G is a cycle of diamonds and contains $4t$ vertices for an odd integer t. In particular, we consider K_4 to be a cycle of one diamond.

Theorem 3. *Let G be a connected 3-regular graph where every vertex is in at least one 3-cycle of G. G has a color-blind coloring if and only if G is not an odd cycle of diamonds. When G is not an odd cycle of diamonds, then* $\mathrm{dal}(G) \leq 3$.

2 Hardness of Computing $\mathrm{dal}(G)$

In this section, we prove Theorem 2 in the standard way. For basics on computational complexity and NP-completeness, see [3]. It is clear that a nondeterministic algorithm can produce and check that a coloring is color-blind distinguishing, so determining $\mathrm{dal}(G) \leq k$ is in NP. We define a polynomial-time reduction[1] that takes a boolean formula in conjunctive normal form where all clauses have three literals and outputs a graph with color-blind index two if and only if the boolean formula is satisfiable.

Theorem 2. *Determining if* $\mathrm{dal}(G) = 2$ *is* NP*-complete, even under the promise that* $\mathrm{dal}(G) \in \{2, 3\}$.

Proof. To prove hardness we will demonstrate a polynomial-time reduction that, given an instance ϕ of 3-SAT, will produce a graph G_ϕ such that $2 \leq \mathrm{dal}(G_\phi) \leq 3$ and such that $\mathrm{dal}(G_\phi) = 2$ if and only if ϕ is satisfiable.

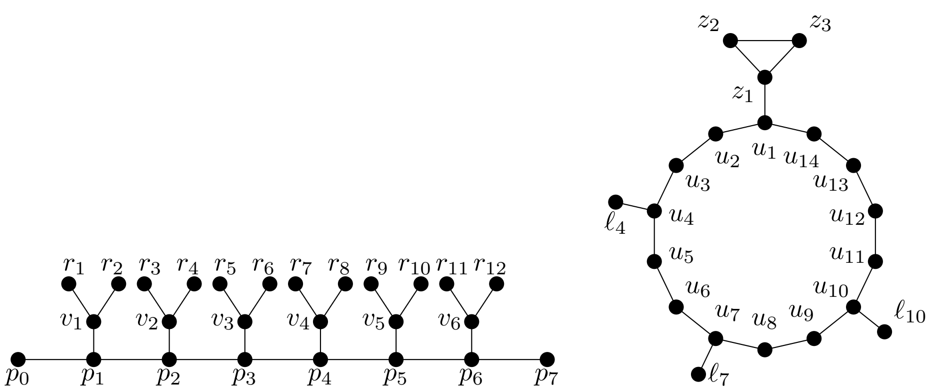
[1] This reduction could easily be implemented in logspace.

Let $\phi(x_1, \dots, x_n) = \bigwedge_{i=1}^{m} C_i$ be a 3-CNF formula with n variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$. Let each clause C_j be given as $C_j = \hat{x}_{i_{j,1}} \vee \hat{x}_{i_{j,2}} \vee \hat{x}_{i_{j,3}}$, where each $\hat{x}_{i_{j,k}}$ is one of $x_{i_{j,k}}$ or $\neg x_{i_{j,k}}$.

We will construct a graph G_ϕ by creating gadgets that represent each variable and clause, and then identifying vertices within those gadgets in order to create G_ϕ. In a 2-edge-coloring of G_ϕ, we consider the color-blind partition $(2,1)$ to be a "true" value while the partition $(3,0)$ corresponds to a "false" value.

Let V be the graph given by vertices $p_0, p_1, \dots, p_{6m+7}, v_1, \dots, v_{6m+6}, r_1, \dots, r_{12m+12}$ where the vertices $p_0 p_1 \dots p_{6m+7}$ form a path, and each v_i is adjacent to p_i, r_{2i-1} and r_{2i}. We will call V the *variable gadget* and create a copy V_i of V for each variable x_i, and list the copy of each vertex w as w^i. The vertices $p_1, \dots, p_{6m+6}$ and $v_1, \dots, v_{6m+6}$ all have degree three, so in a 2-edge-coloring of V, the color-blind partitions take value $(2,1)$ or $(3,0)$. If the color-blind partitions form a proper vertex coloring, then these partitions alternate along the path $p_1 \dots p_{6m+6}$ and along the list $v_1 \dots v_{6m+6}$. Hence, if G_ϕ has a color-blind distinguishing 2-edge-coloring, then the color-blind partition of v_1^i in the copy V_i corresponds to the truth value of x_i. If a clause C_j contains the variable $\hat{x}_i$, the vertices v_{6j+3}^i and v_{6j+4}^i will be used in order to connect the value of x_i or $\neg x_i$ to the clause. First, we must discuss the clause gadgets.

Let L be the graph given by a 3-cycle $z_1 z_2 z_3$, a 14-cycle $u_1 u_2 \dots u_{14}$, and vertices $\ell_4, \ell_7, \ell_{10}$ with the addition of edges $z_1 u_1$, $u_4 \ell_4$, $u_7 \ell_7$, $u_{10} \ell_{10}$. See Figure 1(b) for the graph L. For each clause C_j, create a copy L_j of L and let t_1^j, t_2^j, t_3^j, s_1^j, s_2^j and s_3^j be the copies of the vertices u_4, u_7, u_{10}, ℓ_4, ℓ_7 and ℓ_{10} in L_j.



(a) A variable gadget, V, where $m = 0$.

(b) A clause gadget, L.

Figure 1: Gadgets for variables and clauses.

Claim 2.1. *Let c be a 2-edge-coloring of L and let c^* be the color-blind partitions on the vertices of L. If c^* is a proper vertex coloring, then at least one of the vertices u_4, u_7, and u_{10} has color-blind partition $(2,1)$.*

Proof. Suppose for the sake of contradiction that c^* is a proper vertex coloring and the vertices u_4, u_7, and u_{10} all have color-blind partition $(3,0)$. Thus, the two edges on the cycle incident to one of these vertices have the same color.

In the cycle $z_1 z_2 z_3$, the 2-vertices z_2 and z_3 must have different color-blind partitions. Thus, the edges $z_1 z_2$ and $z_3 z_1$ must receive distinct colors a and b. Thus $c^*(z_1) = (2,1)$ and hence $c^*(u_1) = (3,0)$. Therefore, all 3-vertices on the 14-cycle have the color-blind partition $(3,0)$.

Without loss of generality, let a be the color on the edges $u_1 u_2$ and $u_{14} u_1$. Observe that since the 2-vertices u_2 and u_3 have distinct color-blind partitions, the edge $u_3 u_4$ has color b and hence $u_4 u_5$ has color b. Similarly, observe that the edges $u_6 u_7$ and $u_7 u_8$ have color a, and again that the edges $u_9 u_{10}$ and $u_{10} u_{11}$ have color b.

Now, the 2-vertices u_{11}, u_{12}, u_{13}, and u_{14} should have distinct color-blind partitions, but since the color of $u_{10} u_{11}$ is b and the color of $u_{14} u_1$ is a, this is impossible. □

It remains to show that if at least one of these vertices has color-blind partition $(2,1)$, then we can give a color-blind distinguishing 2-edge-coloring to the gadget L.

Claim 2.2. *Let p_4, p_7, p_{10} be three partitions in $\{(2,1),(3,0)\}$. If at least one value p_i is $(2,1)$, then there exists a 2-edge-coloring c of L such that c^* is a proper vertex coloring and $c^*(u_4) = p_4$, $c^*(u_7) = p_7$, and $c^*(u_{10}) = p_{10}$.*

Proof. Select $j \in \{4,7,10\}$ such that $p_j = (2,1)$. We construct a 2-edge-coloring c of L by first coloring the edges v_1v_2, v_2v_3, u_1v_1, and u_1u_2 with color a and the edge v_1v_3 with color b. We will color the cycle $u_1u_2\dots u_{14}$ by coloring the edges of the paths $u_1u_2u_3u_4$, $u_4u_5u_5u_7$, $u_7u_8u_9u_{10}$, and $u_{10}u_{11}u_{12}u_{13}u_{14}u_1$ in a way that ensures that the 2-vertices are properly colored. When we reach each 3-vertex u_k, we will color the edge u_ku_{k+1} using the same color as $u_{k-1}u_k$ unless $k = j$, in which case we color u_ju_{j+1} the opposite color as $u_{j-1}u_j$. Color the edges $u_k\ell_k$ such that the color-blind partition on u_k is equal to p_k. Since the edge pairs u_1u_2 and u_3u_4, u_4u_5 and u_6u_7, u_7u_8 and u_9u_{10} must receive opposite colors, observe that the edge $u_{10}u_{11}$ will have color a using this coloring. Also observe that the edge pair $u_{10}u_{11}$ and $u_{14}u_1$ receive the same color, so the vertex u_1 has color-blind partition $(3,0)$ and hence we have the desired coloring c. □

We are now prepared to define G_ϕ. First, create all copies V_i of the variable gadget V for all variables x_i. Then create all copies L_j of the clause gadget L for all clauses C_j. Finally, consider each variable $\hat{x}_{i_{j,k}}$ in each clause C_j. If $\hat{x}_{i_{j,k}} = x_{i_{j,k}}$, then identify the vertex $v_{6j+3}^{i_{j,k}}$ with t_k^j, and identify r_{12j+5} and r_{12j+6} with the 2-vertices adjacent to t_k^j, and $p_{6j+3}^{i_{j,k}}$ with the leaf s_k^j. If $\hat{x}_{i_{j,k}} = \neg x_{i_{j,k}}$, then identify the vertex $v_{6j+4}^{i_{j,k}}$ with t_k^j, and identify r_{12j+7} and r_{12j+8} with the 2-vertices adjacent to t_k^j, and $p_{6j+4}^{i_{j,k}}$ with the leaf s_k^j.

Let c be a 2-edge-coloring of G_ϕ and define the variable assignment $x_i = \begin{cases} \text{true} & c^*(v_1^i) = (2,1) \\ \text{false} & c^*(v_1^i) = (3,0) \end{cases}$. Observe that if c^* is a proper vertex coloring, then $c^*(v_{6j+3}^i) = c^*(v_1^i)$ and $c^*(v_{6j+4}^i) \neq c^*(v_1^i)$ for each variable gadget V_i and each clause gadget L_j. Then, since c^* is a proper vertex coloring, Claim 2.1 implies that one of the vertices u_4, u_7, u_{10} in each clause gadget L_j has color-blind partition $(2,1)$ and therefore the clause is satisfied by the variable assignment. Therefore, if $\operatorname{dal}(G_\phi) = 2$, then ϕ is satisfiable.

In order to demonstrate that every satisfiable assignment corresponds to a color-blind 2-edge-coloring of G_ϕ, we use the following claim.

Claim 2.3. *Let V_i be a variable gadget and fix $j \in \{1,\dots,m\}$ and $t \in \{3,4\}$. Let D be the subgraph induced by the vertices $p_{6j+1}, p_{6j+2}, \dots, p_{6j+7}$, v_{6j+3}, v_{6j+4}, and their neighbors. Let c be an assignment of the colors $\{1,2\}$ to the edges incident to p_{6j+1} and v_{6j+t} such that $c^*(p_{6j+1}) \neq c^*(v_{6j+3})$ when $t = 3$ and $c^*(p_{6j+1}) = c^*(v_{6j+4})$ when $t = 4$. There exists a 2-edge-coloring c' of the remaining edges such that $(c \cup c')^*$ is a proper vertex coloring of D.*

Claim 2.3 follows by exhaustive enumeration of the possible colorings of the graph D, so the proof is omitted[2].

Let $x_1,\dots,x_n$ be a variable assignment such that $\phi(x_1,\dots,x_n)$ is true. For each clause C_j, there is at least one variable $\hat{x}_{i_{j,k}}$ that is true, so by Claim 2.2 there exists a 2-edge-coloring c_j of K_j where c_j^* is a proper vertex coloring and the color-blind partitions of u_4, u_7 and u_{10} correspond to the truth values of $\hat{x}_{i_{j,1}}$, $\hat{x}_{i_{j,2}}$, and $\hat{x}_{i_{j,3}}$, respectively. Fix a 2-edge-coloring of each vertex v_1^i such that the color-blind partition at v_1^i corresponds to the truth value of x_i. Finally, by Claim 2.3 these 2-edge-colorings of the vertices $v_1^1,\dots,v_1^n$ and clause gadgets $L_1,\dots,L_m$ extend to a 2-edge-coloring c of G_ϕ where c^* is a proper vertex coloring.

Thus, determining if $\operatorname{dal}(G_\phi) \le 2$ is NP-hard.

We complete our investigation by demonstrating that $\operatorname{dal}(G_\phi) \le 3$ always. To generate a 3-edge-coloring of G_ϕ, fix a variable assignment $x_1,\dots,x_n$. If a clause C_j is satisfied by this variable assignment, then use

[2]The algorithm for enumerating all colorings is available as a Sage worksheet at `http://orion.math.iastate.edu/dstolee/r/cbindex.htm`

Claim 2.2 to find a 2-edge-coloring on the clause gadget L_j. If a clause C_j is not satisfied by this variable assignment, then assign color 1 to the edge set

$$\{z_1z_2, z_2z_3, u_1z_1, u_2u_3, u_3u_4, u_4\ell_4, u_4u_5, u_5u_6, u_9u_{10}, u_{10}\ell_{10}, u_{10}u_{11}, u_{11}u_{12}\},$$

assign color 2 to the edge set

$$\{z_1z_3, u_1u_2, u_6u_7, u_7\ell_7, u_7u_8, u_8u_9, u_{12}u_{13}, u_{13}u_{14}\},$$

and finally assign color 3 to the edge u_14u_1. Observe that this coloring is color-blind distinguishing on L_j with $c^*(u_4) = c^*(u_7) = c^*(u_{10}) = (3, 0)$. Using Claim 2.3, this coloring extends to the variable gadgets and hence there is a color-blind distinguishing 3-edge-coloring of G_ϕ. □

In the next sections, we explore determining the color-blind index of graphs using properties that avoid the constructions in the above reduction from 3-SAT.

3 Regular Bipartite Graphs and 2-Colorable Hypergraphs

In Section 2, we demonstrated that it is NP-complete to determine if $\mathrm{dal}(G) = 2$, even when promised that $\mathrm{dal}(G) \in \{2, 3\}$. One particular instance of this situation is in the case of regular bipartite graphs, as Kalinowski, Pilśniak, Przybyło, Woźniak [9] determined an upper bound on the color-blind index.

Theorem 4 (Kalinowski, Pilśniak, Przybyło, Woźniak [9])**.** *If G is a k-regular bipartite graph with $k \geq 2$, then* $\mathrm{dal}(G) \leq 3$.

We demonstrate that when G is a k-regular bipartite graph, $\mathrm{dal}(G) = 2$ if and only if at least one of two corresponding k-regular, k-uniform hypergraphs is 2-colorable. Erdős and Lovász [6] implicitly proved that k-regular k-uniform hypergraphs are 2-colorable for all $k \geq 9$ in the first use of the Lovász Local Lemma. Several results [2, 5, 7, 14] proved different cases for $k < 9$ and also demonstrated that some 3-regular 3-uniform hypergraphs are not 2-colorable, such as the Fano plane. Thomassen [13] implicitly proved the general case, and Henning and Yeo [8] proved it explicitly.

Theorem 5 (Thomassen [13], Henning and Yeo [8])**.** *Let $k \geq 4$. If $\mathcal{H}$ is a k-regular k-uniform hypergraph, then $\mathcal{H}$ is 2-colorable.*

McCuaig [10] has a characterization of 3-regular, 3-uniform, 2-colorable hypergraphs when the 2-vertex-coloring is forced to be *balanced*. A general characterization is not known for 3-regular, 3-uniform, 2-colorable hypergraphs.

If $\mathcal{H}$ is a k-uniform hypergraph with vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H})$, the *vertex-edge incidence graph* of $\mathcal{H}$ is the bipartite graph G with vertex set $V(\mathcal{H}) \cup E(\mathcal{H})$ where a vertex $v \in V(\mathcal{H})$ and edge $e \in E(\mathcal{H})$ are incident in G if and only if $v \in e$. Note that since $\mathcal{H}$ is k-uniform, all of the vertices in the $E(\mathcal{H})$ part of G have degree k; G is k-regular if and only if $\mathcal{H}$ is also k-regular and k-uniform.

Proposition 6. *Let $\mathcal{H}$ be a k-uniform hypergraph and G its vertex-edge incidence graph. If $\mathcal{H}$ is 2-colorable, then* $\mathrm{dal}(G) \leq 2$.

Proof. Let $V = V(\mathcal{H})$ and $E = E(\mathcal{H})$, and let G be the bipartite vertex-edge incidence graph with vertex set $V \cup E$. Let $c : V \to \{1, 2\}$ be a proper 2-vertex-coloring of $\mathcal{H}$. For each $v \in V$, and edge $e \in E$ where $v \in e$, let the edge ve of G be colored $c(ve) = c(v)$. Let c^* be the color-blind partition on the vertices of G induced by the coloring on the edges of G. The color-blind partition at every vertex $v \in V$ is $c^*(v) = (d_{\mathcal{H}}(v), 0)$. Since c is a proper 2-vertex-coloring of $\mathcal{H}$, the color-blind partition at every edge $e \in E$ is $c^*(e) = (k - i, i)$, where $1 \leq i \leq \lfloor k/2 \rfloor$. Therefore, c^* is a proper vertex coloring of G and $\mathrm{dal}(G) \leq 2$. □

If $G = (X \cup Y, E)$ is a k-regular bipartite graph, then there are two (possibly isomorphic) k-uniform hypergraphs $\mathcal{H}_X, \mathcal{H}_Y$, defined by $V(\mathcal{H}_X) = X$ and $E(\mathcal{H}_X) = \{N_G(y) : y \in Y\}$, $V(\mathcal{H}_Y) = Y$ and $E(\mathcal{H}_Y) = \{N_G(x) : x \in X\}$. When $k \geq 4$ and G is a k-regular bipartite graph, then both $\mathcal{H}_X$ and $\mathcal{H}_Y$ are 2-colorable by the theorem of Henning and Yeo [8].

Proposition 7. *If $G = (X \cup Y, E)$ is a connected 3-regular bipartite graph with $\text{dal}(G) \leq 2$, then at least one of the 3-regular, 3-uniform hypergraphs $\mathcal{H}_X$ or $\mathcal{H}_Y$ is 2-colorable.*

Proof. Let $c : E(G) \to \{1, 2\}$ be a 2-edge-coloring of G such that $c^* : V(G) \to \{(3, 0), (2, 1)\}$ is a proper vertex coloring of G. Then, exactly one of X or Y has all vertices colored with $(3, 0)$ and the other is colored with $(2, 1)$, since G is connected. Thus, at least one of $\mathcal{H}_X$ or $\mathcal{H}_Y$ has a 2-vertex-coloring where $c(v)$ is the unique color on the edges of G incident to v, and this coloring is proper since each edge is incident to two vertices with one color and one vertex with the other. □

Since regular bipartite graphs are well understood, but the existence of a color-blind coloring in a general cubic graph is not well understood, our next section investigates cubic graphs that are as far from being bipartite as possible.

4 Cubic Graphs with Many 3-cycles

In this section, we prove Theorem 3 concerning 3-regular graphs where every vertex is in at least one 3-cycle. We first demonstrate the case where G has no color-blind coloring.

Lemma 8. *If G is an odd cycle of diamonds, then* $\text{dal}(G) = \infty$.

Proof. Suppose for the sake of contradiction that c is a color-blind distinguishing k-edge-coloring of G for some k and note that c^* takes the colors $(3, 0, 0), (2, 1, 0)$, and $(1, 1, 1)$. For every diamond $xyzw$ where xyz and yzw are 3-cycles, observe that $c^*(x) = c^*(w)$. Thus, for every diamond, we can associate the the diamond with the c^*-color of the endpoints. Since c^* is proper, adjacent diamonds must receive distinct colors. Since an odd cycle is not 2-colorable, there must be a diamond $xyzw$ with endpoints colored $(3, 0, 0)$. Then the edges xy and xz and the edges wy and wz receive the same colors under c. So regardless of $c(yz)$, we must have $c^*(y) = c^*(z)$. Hence c^* is not proper. □

We prove Theorem 3 by using a strengthened induction, presented in Theorem 9. A *$\{1, 3\}$-regular graph* is a graph where every vertex has degree 1 or 3.

Theorem 9. *Let G be a connected $\{1, 3\}$-regular graph where every 1-vertex is adjacent to a 3-vertex and every 3-vertex is in at least one 3-cycle. There exists a color-blind distinguishing 3-edge-coloring of G if and only if G is not an odd cycle of diamonds. When a color-blind distinguishing 3-edge-coloring exists, if v is a 3-vertex adjacent to a 1-vertex, then there are two color-blind distinguishing 3-edge-colorings c_1, c_2 where $c_1^*(v) \neq c_2^*(v)$.*

Proof. Among examples of graphs that satisfy the hypothesis but do not have color-blind distinguishing 3-edge-colorings, select G to minimize $n(G) + e(G)$. We shall prove that G is either a subgraph of a small list of graphs that contain color-blind distinguishing 3-edge-colorings, or contains one of a small list of *reducible configurations.*

Figure 2 lists four graphs and demonstrates color-blind distinguishing 3-edge-colorings that satisfy the theorem statement. Therefore, G is not among this list.

Claim 9.1. *G does not contain a cut-edge uv where $d(u) = d(v) = 3$.*

Proof. Suppose uv is a cut-edge with $d(u) = d(v) = 3$. Let G_1 and G_2 be the components of $G - uv$ where $u \in V(G_1)$ and $v \in V(G_2)$, and let $G'_i = G_i + uv$ for each $i \in \{1, 2\}$. Observe that $n(G'_i) + e(G'_i) < n(G) + e(G)$. Also, neither is an odd cycle of diamonds, as they have vertices of degree 1. Therefore, there are color-blind distinguishing 3-edge-colorings $c_1 : E(G'_1) \to \{1, 2, 3\}$ and $c_2 : E(G'_2) \to \{a, b, c\}$ such that $c_2^*(v) \neq c_1^*(u)$. The color set $\{a, b, c\}$ can be permuted to $\{1, 2, 3\}$ such that $c_2(uv)$ is mapped to $c_1(uv)$. Under this permutation, c_1 and c_2 combine to form a color-blind distinguishing 3-edge-coloring of G. □

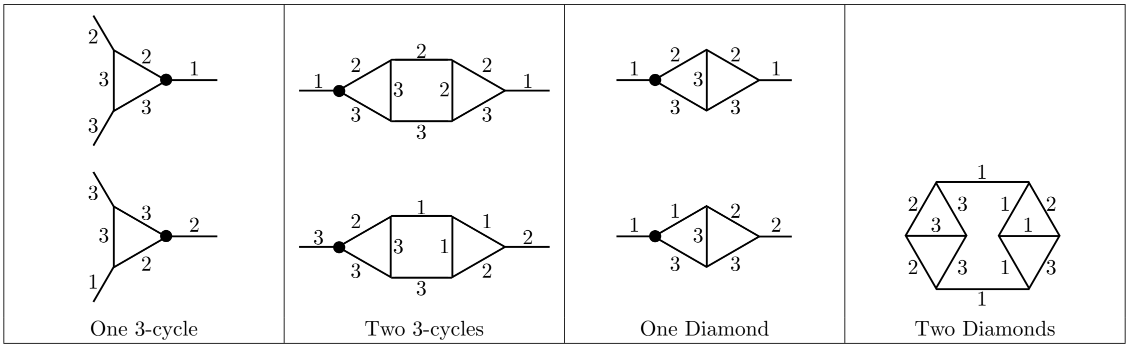


Figure 2: Base case graphs and their color-blind distinguishing 3-edge-coloring(s) where the 3-vertex v is highlighted.

Further note that every color-blind distinguishing 3-edge-coloring of G_1' extends to a color-blind distinguishing 3-edge-coloring of G, and by symmetry every color-blind distinguishing 3-edge-coloring of G_2' extends to a color-blind distinguishing 3-edge-coloring of G. Thus, for any vertex x of degree 3 adjacent to a vertex of degree 1, x is also a vertex of degree 3 adjacent to a vertex of degree 1 in some G_i' and hence has distinct color-blind partitions for two colorings in that G_i'. These colorings both extend to G, so the two distinct color-blind partitions on x also appear in color-blind distinguishing 3-edge-colorings of G.

Definition (Reducible Configurations)**.** Let H be a $\{1,3\}$-regular graph and $D \subset V(H)$ such that for every $v \in D$, there is at most one vertex $u = u(v) \in N_H(v) \setminus D$. Let H_D be the subgraph of H induced by D, let S be the set of neighbors of D that are not in D. Let M be a matching that saturates S, using edges in the edge cut $[D, S]$ or using pairs from S.

Let $c : M \to \{1, 2, 3\}$ and $c^* : S \to \{(3, 0, 0), (2, 1, 0), (1, 1, 1)\}$ be assignments such that $c^*(u) \neq c^*(v)$ for all edges $uv \in M$. For an edge $xy \in [D, S]$ where $x \in D$ and $y \in S$, define $c(xy)$ to be $c(yz)$ where yz is the edge of M covering y. Such a pair (c, c^*) is a *potential pair*.

The triple (H, D, M) is a *reducible configuration* if $E(H_D) \neq \varnothing$, and for every potential pair (c, c^*), there exists an extension of c to include $E(H_D)$ where the color-blind partitions for vertices in D create an extension of c^* to D that is a proper vertex coloring of $D \cup S$. We say that a graph G *contains* a reducible configuration (H, D, M) if it contains H as a subgraph, and the corresponding vertices of S in that subgraph have degree 3 in G.

Figure 3 contains a list of four reducible configurations. Some of these are checkable by hand, while others were verified to be reducible using a computer[3]. If (H, D, M) is a reducible configuration and $H \subseteq G$, we use $G - D + M$ to denote the *reduced graph* given by deleting the edges with at least one endpoint in D, adding the edges in M, and removing any isolated vertices. Observe that for every reducible configuration in Figure 3, every vertex x in G (not in D) that has degree 3 and is adjacent to a vertex of degree 1 remains a vertex of this type in the reduced graph $G - D + M$. Therefore, the two colorings that provide distinct color-blind partitions for x in $G - D + M$ each extend to a color-blind distinguishing 3-edge-coloring of G.

Claim 9.2. *Let H be the graph in the 1-diamond reduction, with $D = \{a, b, c, \}$ and $M = \{pq, wr\}$. G does not contain the reducible configuration (H, D, M).*

Proof. Suppose G contains H as a subgraph. Let $G' = G - D + M$, and observe that $n(G') + e(G') < n(G) + e(G)$. Also, by Claim 9.1 the edge cx is not a cut-edge of G, G' is connected and is not an odd cycle of diamonds. Therefore, there exists a color-blind distinguishing 3-edge-coloring $c : E(G') \to \{1, 2, 3\}$ where c^* is a proper vertex coloring on G' and hence a proper vertex coloring on M. By the definition of reducible configuration, this coloring c extends to a color-blind distinguishing 3-edge-coloring of G, a contradiction. □

[3]The algorithm for checking reducibility is available as a Sage worksheet at `http://orion.math.iastate.edu/dstolee/r/cbindex.htm`

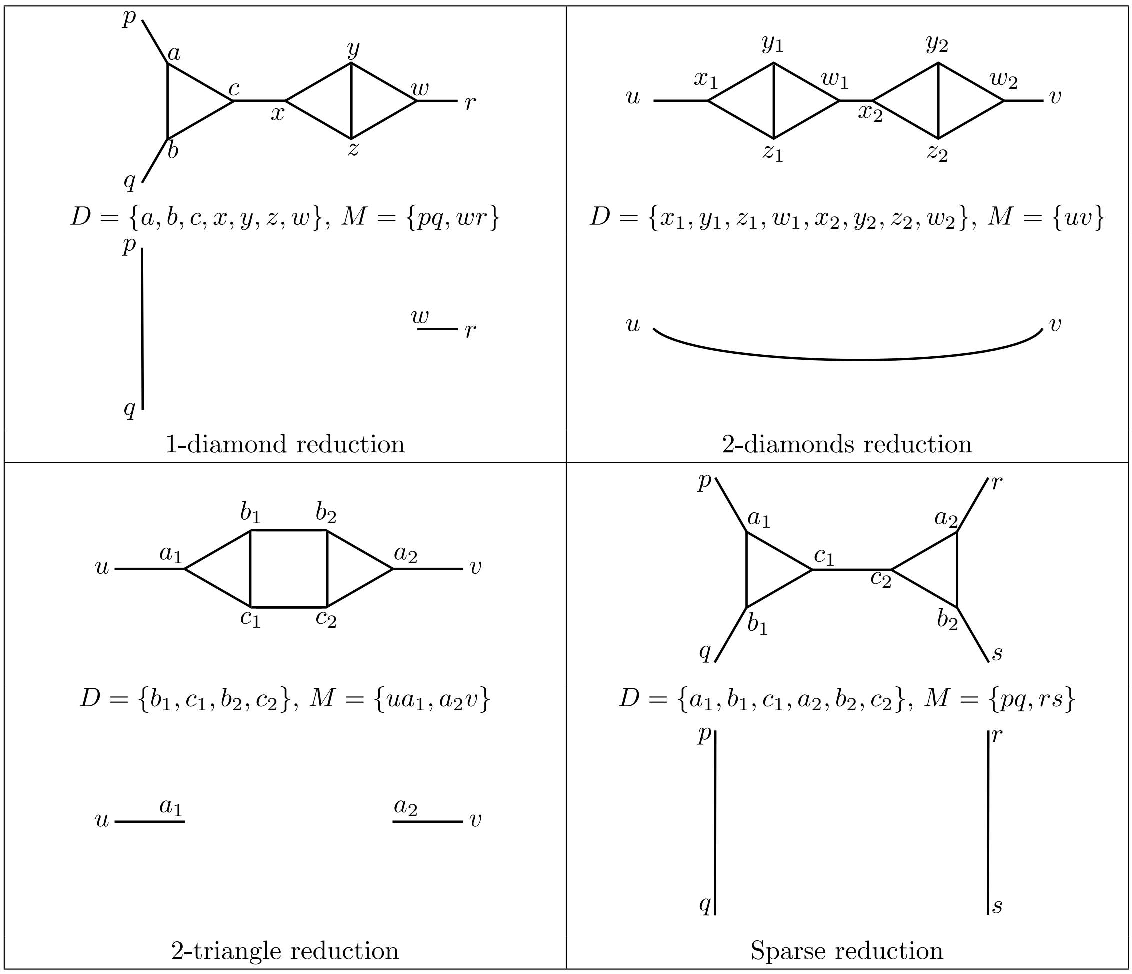


Figure 3: The Reducible Configurations and their Reductions.

Claim 9.3. *Let H be the graph in the 2-diamond reduction, with $D = \{x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2\}$ and $M = \{uv\}$. G does not contain the reducible configuration (H, D, M).*

Proof. Suppose G contains H as a subgraph. Let $G' = G - D + M$, and observe that $n(G') + e(G') < n(G) + e(G)$. Also, since G is not an odd cycle of diamonds, G' is not an odd cycle of diamonds. Therefore, there exists a color-blind distinguishing 3-edge-coloring $c : E(G') \to \{1, 2, 3\}$ where c^* is a proper vertex coloring on G' and hence a proper vertex coloring on M. By the definition of reducible configuration, this coloring c extends to a color-blind distinguishing 3-edge-coloring of G, a contradiction. □

Claim 9.4. *Let H be the graph in the 2-triangle reduction, with $D = \{b_1, c_1, b_2, c_2\}$ and $M = \{ua_1, a_2v\}$. G does not contain the reducible configuration (H, D, M).*

Proof. Suppose G contains H as a proper subgraph. Since G is connected, at least one of u and v is a 3-vertex; without loss of generality u is a 3-vertex. The edge ua_1 is not a cut-edge, by 9.1, so v is also a 3-vertex. Let $G' = G - D + M$, and observe that $n(G') + e(G') < n(G) + e(G)$. Observe that G' is connected, $\{1, 3\}$-regular and not an odd cycle of diamonds, and that every 1-vertex is adjacent to a 3-vertex. Therefore, there exists a color-blind distinguishing 3-edge-coloring $c : E(G') \to \{1, 2, 3\}$ where c^* is a proper vertex coloring on G' and hence a proper vertex coloring on M. By the definition of reducible configuration, this coloring c extends to a color-blind distinguishing 3-edge-coloring of G, a contradiction. □

Claim 9.5. *Let H be the graph in the sparse reduction, with $D = \{a_1, b_1, c_1, a_2, b_2, c_2\}$ and $M = \{pq, rs\}$. G does not contain the reducible configuration (H, D, M).*

Proof. Suppose G contains H as a subgraph. Let $G' = G - D + M$, and observe that $n(G') + e(G') < n(G) + e(G)$. Since G does not contain the 2-triangle reduction, pq and rs are not edges of G and hence G' is a $\{1, 3\}$-regular graph. Also, since the edge c_1c_2 is not a cut-edge and since G does not contain the 1-diamond reduction, G' is connected and is not an odd cycle of diamonds. Therefore, there exists a color-blind distinguishing 3-edge-coloring $c : E(G') \to \{1, 2, 3\}$ where c^* is a proper vertex coloring on G' and hence a proper vertex coloring on M. By the definition of reducible configuration, this coloring c extends to a color-blind distinguishing 3-edge-coloring of G, a contradiction. □

We complete our proof by demonstrating that G contains one of the reducible configurations.

Suppose that G contains a diamond $xyzw$ where $xyzw$ is a 4-cycle and yz is an edge. Since G is not a single diamond, without loss of generality we have that the vertex adjacent to x, say u, is in a 3-cycle or a diamond. Therefore, G is isomorphic to or contains either the 1-diamond reduction or the 2-diamonds reduction. We may now assume that G does not contain any diamond.

Let abc be a 3-cycle in G. Since G has more than one 3-cycle, at least one vertex is adjacent to a vertex in another 3-cycle. If two vertices in $\{a, b, c\}$ are adjacent to the same 3-cycle, then G is isomorphic to or contains the 2-triangle reduction. Therefore, we may assume that every pair of adjacent 3-cycles have exactly one edge between them. However, a pair of adjacent 3-cycles and their neighboring vertices form a sparse reduction as a subgraph of G.

Therefore, the minimal counterexample G does not exist and the theorem holds. □

Remark. The use of reducible configurations demonstrates a polynomial-time algorithm for finding a color-blind distinguishing 3-edge-coloring of a cubic graph where every vertex is in exactly one 3-cycle. The algorithm works recursively, with base cases among the list of two diamonds or two 3-cycles where the two color-blind distinguishing 3-edge-colorings c_1 and c_2 can be produced in constant time. The algorithm first searches for a cut-edge uv with $d(u) = d(v) = 3$ and if one exists creates the graphs G_1' and G_2' as in Claim 9.1; recursion on these graphs produces colorings that can be combined to form a coloring of G. If no such cut-edge is found, then the algorithm searches for one of the reducible configurations, one of which will exist. By performing the reduction from G to $G - D + M$ as specified by the reducible configuration, the algorithm can recursively find a coloring on $G - D + M$ and in constant time produce an extension to G.

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