Automorphism Groups and Adversarial Vertex Deletions

Derrick Stolee
Department of Mathematics
Department of Computer Science
Iowa State University
dstolee@iastate.edu

November 25, 2013

Abstract

Any finite group can be encoded as the automorphism group of an unlabeled simple graph. Recently Hartke, Kolb, Nishikawa, and Stolee (2010) demonstrated a construction that allows any ordered pair of finite groups to be represented as the automorphism group of a graph and a vertex-deleted subgraph. In this note, we describe a generalized scenario as a game between a player and an adversary: An adversary provides a list of finite groups and a number of rounds. The player constructs a graph with automorphism group isomorphic to the first group. In the following rounds, the adversary selects a group and the player deletes a vertex such that the automorphism group of the corresponding vertex-deleted subgraph is isomorphic to the selected group. We provide a construction that allows the player to appropriately respond to any sequence of challenges from the adversary.

Automorphisms of graphs are incredibly unstable. The slightest perturbation of the graph can greatly change the automorphism group. In this note, we show there exist graphs whose automorphism groups can change dramatically under certain sequences of vertex deletions. We consider undirected, unlabeled, and simple graphs, denoted F, G, or H, and finite groups, denoted Γ. The automorphism group of a graph G is denoted $\mathrm{Aut}(G)$.

Frucht [3] proved that graphs have the ability to encode the structure of any finite group.

Theorem 1 (Frucht [3])**.** *Let Γ be a finite group. There exists a graph G with $\mathrm{Aut}(G) \cong \Gamma$.*

Hartke, Kolb, Nishikawa, and Stolee [4] proved that any ordered pair of finite groups can be represented by a graph and a vertex-deleted subgraph. Their work was motivated by consequences to the Reconstruction Conjecture (see Bondy [2]) and isomorph-free generation (see McKay [5]).

Theorem 2 (Hartke, Kolb, Nishikawa, Stolee [4])**.** *Let Γ_0 and Γ_1 be finite groups. There exists a graph G and a vertex $v \in V(G)$ such that $\mathrm{Aut}(G) \cong \Gamma_0$ and $\mathrm{Aut}(G - v) \cong \Gamma_1$.*

There are two natural extensions of this process to a sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$ of finite groups using two types of vertex deletions: single deletions or iterated deletions.

Question. *Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$ be finite groups. Does there exist a graph G with vertices $v_1, \ldots, v_k \in V(G)$ such that* $\mathrm{Aut}(G) \cong \Gamma_0$ *and for all $i \in \{1, \ldots, k\}$,*

1. *(Single Deletions)* $\mathrm{Aut}(G - v_i) \cong \Gamma_i$*?*
2. *(Iterated Deletions)* $\mathrm{Aut}(G - v_1 - \cdots - v_i) \cong \Gamma_i$*?*

In fact, both of these types of deletions can be combined in an even more general situation, posed as the *vertex deletion game* between a player and an adversary:

The Vertex Deletion Game

Round 0:

Adversary: Selects finite groups $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$, and a number $\ell \geq 1$.

Player: Constructs a graph G_0 with $\mathrm{Aut}(G_0) \cong \Gamma_0$.

Round j: $(1 \leq j \leq \ell)$

Adversary: Selects a group $\Gamma_{i_j} \in \{\Gamma_1, \ldots, \Gamma_k\}$.

Player: Selects a vertex $v_j \in V(G_{j-1})$, defines $G_j = G_{j-1} - v_j$, and asserts $\mathrm{Aut}(G_j) \cong \Gamma_{i_j}$.

Note that this game generalizes both single deletions (play the game with $\ell = 1$) and iterated deletions (play the game with $\ell = k$, and the adversary selects $\Gamma_{i_j} = \Gamma_j$ for all $j \in \{1, \ldots, k\}$). By carefully constructing G_0, the player can survive ℓ rounds against the adversary.

Theorem 3 (Adversarial Iterated Deletions)**.** *Suppose the adversary selects Γ_0, Γ_1, $\ldots$, Γ_k as finite groups and integer $\ell \geq 1$ in Round 0. The player can construct a graph G_0 with* $\mathrm{Aut}(G_0) \cong \Gamma_0$ *so that the assertions* $\mathrm{Aut}(G_j) \cong \Gamma_{i_j}$ *hold for all ℓ remaining rounds.*

Instead of using the vertex deletion game, there is an equivalent statement of the previous theorem using a sequence of alternating quantifiers.

Theorem 4 (Adversarial Iterated Deletions; alternate form)**.** *For all numbers $k, \ell \geq 1$ and finite groups Γ_0, Γ_1, $\ldots$, Γ_k, there exists a graph G_0 such that* $\mathrm{Aut}(G_0) \cong \Gamma_0$ *and*

$$\forall i_1\, \exists v_1\, \forall i_2\, \exists v_2\, \cdots \forall i_\ell\, \exists v_\ell\, \forall j,\ \mathrm{Aut}(G_0 - v_1 - \cdots - v_j) \cong \Gamma_{i_j},$$

where the domain of j is $\{1, \ldots, \ell\}$, the domain of each i_j is $\{1, \ldots, k\}$, and the domain of each v_j is $V(G_0) \setminus \{v_1, \ldots, v_{j-1}\}$.

A group is *trivial* if it consists only of the identity element. For a graph G and vertex $v \in V(G)$, the *stabilizer* of v in G, denoted $\mathrm{Stab}_G(v)$, is the subgroup of $\mathrm{Aut}(G)$ given by permutations τ where $\tau(v) = v$.

Our starting point is the following lemma from [4].

Lemma 5 (Hartke, Kolb, Nishikawa, Stolee [4, Lemma 2.2])**.** *For any finite group Γ, there is a connected graph G and a vertex $v \in V(G)$ where* $\mathrm{Aut}(G) \cong \Gamma$ *and* $\mathrm{Stab}_G(v)$ *is trivial.*

We now describe a gadget which will be used to build the full construction for Theorem 3.

Lemma 6. *Let Γ be a finite group. There exists a graph H and two vertices $x, y \in V(H)$ so that* $\mathrm{Aut}(H)$ *is trivial, $H - x$ is connected,* $\mathrm{Aut}(H - x) \cong \Gamma$*, and* $\mathrm{Stab}_{H-x}(y)$ *is trivial.*

Proof. By Lemma 5, there exists a connected graph G and a vertex $y \in V(G)$ so that $\mathrm{Aut}(G) \cong \Gamma$ and $\mathrm{Stab}_G(y)$ is trivial. Let $n = |V(G)|$ and order the vertices of G as $V(G) = \{v_1, \dots, v_n\}$ and $v_1 = y$.

Let H be a graph with vertex set $V(H) = \{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\} \cup \{x, z, w\}$. The graph H has an edge $v_i v_j$ if and only if that edge is present in G. For every $j \in \{1, \dots, n\}$, the pair $u_j v_j$ is an edge. The vertex z is adjacent to all vertices v_j for $j \in \{1, \dots, n\}$. The vertex x is adjacent to z, all vertices v_j for $j \in \{1, \dots, n\}$ and adjacent to the vertices u_i for $i \in \{2, \dots, n\}$. Finally, the vertex w is adjacent only to x and z.

The only vertex of degree 1 in H is u_1, so every automorphism of H stabilizes u_1 and thus also stabilizes v_1. All vertices $v_1, \dots, v_n$ have degree at least three and degree at most $n+1$. The vertex x is the only vertex of degree $2n+1$, so every automorphism of H stabilizes x. The vertex z is the only vertex of degree $n+2$, so every automorphism of H stabilizes z and hence also stabilizes w. Other than w, the vertices $u_2, \dots, u_n$ are the only vertices of degree 2, so every automorphism set-wise stabilizes $\{u_2, \dots, u_n\}$. This implies that every automorphism of H set-wise stabilizes $\{v_1, \dots, v_n\}$ and hence restricting an automorphism of H to $V(G)$ induces an automorphism of G. However, every automorphism point-wise stabilizes v_1. Since $v_1 = y$ and $\mathrm{Stab}_G(y)$ is trivial, every automorphism of H must point-wise stabilize $V(G)$. Thus the vertices $u_1, \dots, u_n$ are also point-wise stabilized and the automorphism group of H is trivial.

Now consider $H' = H - x$. The vertex z is the only vertex of degree $n+1$, so every automorphism of H' stabilizes z and w. Other than w, the only vertices of degree 1 are $u_1, \dots, u_n$, so every automorphism of H' set-wise stabilizes $\{u_1, \dots, u_n\}$. Since the vertices $u_1, \dots, u_n$ are adjacent only to vertices in $V(G)$, every automorphism of H' set-wise stabilizes $V(G)$. Hence, every automorphism of H' restricted to $V(G)$ is an automorphism of G. Observe that every automorphism $\sigma \in \mathrm{Aut}(G)$ extends to an automorphism of H by assigning $\sigma(u_i) = u_j$ whenever $\sigma(v_i) = v_j$. Thus, $\mathrm{Aut}(H - x) \cong \mathrm{Aut}(G) \cong \Gamma$.

Since the automorphisms of $H - x$ correspond directly to automorphisms of G, observe that $\mathrm{Stab}_{H-x}(y)$ is trivial. □

We are now sufficiently prepared to prove the main theorem. The gadget from Lemma 6 has two purposes:

1. "Reveal" symmetry: When x is deleted, the automorphism group Γ is revealed.

2. "Remove" symmetry: When y is stabilized within $H - x$, all non-trivial automorphisms of $H - x$ are removed.

Our construction for the graph G_0 carefully places many copies of this gadget in such a way that the player has access to a "revealing" vertex (x) that simultaneously stabilizes the "removing" vertex (y) in the previous gadget. Therefore, we have a sequence of deletions which remove all previous symmetry and reveal only the requested symmetry.

Proof of Theorem 3. Note that the case $k = \ell = 1$ holds by Theorem 2. We assume that the groups $\Gamma_1, \dots, \Gamma_k$ are distinct with respect to isomorphism.

By Lemma 6, for every $i \in \{0, 1, \dots, k\}$ there is a graph H_i with vertices $x_i, y_i \in V(H_i)$ such that $\mathrm{Aut}(H_i)$ is trivial, $\mathrm{Aut}(H_i - x_i) \cong \Gamma_i$, and $\mathrm{Stab}_{H_i - x_i}(y_i)$ is trivial. For all $i \in \{0, \dots, k\}$, let O_i be the orbit of y_i in $H_i - x_i$. Since the groups $\Gamma_1, \dots, \Gamma_k$ are pairwise non-isomorphic, then by the construction of Lemma 6 the graphs $H_0, H_1, \dots, H_k$ and $H_0 - x_0, \dots, H_k - x_k$ are all pairwise non-isomorphic. Also by the construction of Lemma 6, no graph H_i or $H_i - x_i$ has a dominating vertex.

We construct the graph G_0 by building graphs $F_0, F_1, \dots, F_\ell$ iteratively. Let F_0 be the graph given by taking $H_0 - x_0$ and adding vertices a_0, b_0 where a_0 is adjacent to all vertices in $H_0 - x_0$ and b_0 is adjacent to only a_0. Let $U_0 = O_0$.

For all $j \in \{1, \dots, \ell\}$, we will build F_j by adding vertices and edges to F_{j-1}. During the process, F_{j-1} will remain an induced subgraph of F_j. For all vertices $v \in U_{j-1}$ and $i \in \{1, \dots, k\}$, add a copy $H_i^{(j,v)}$ of H_i to F_{j-1} and add edges from v to each vertex of $H_i^{(j,v)}$. Let $x_i^{(j,v)}$ and $y_i^{(j,v)}$ denote the copies of x_i and y_i in $H_i^{(j,v)}$. Let $O_i^{(j,v)}$ be the copy of O_i within $H_i^{(j,v)}$ and define $U_j = \cup_{v \in U_{j-1}} \cup_{i=1}^{k} O_i^{(j,v)}$. Add vertices a_j, b_j where a_j is adjacent to all vertices in $V(F_j) \setminus V(F_{j-1})$ and the vertices a_{j-1} and b_j. Figure 1 shows a visualization of this construction.

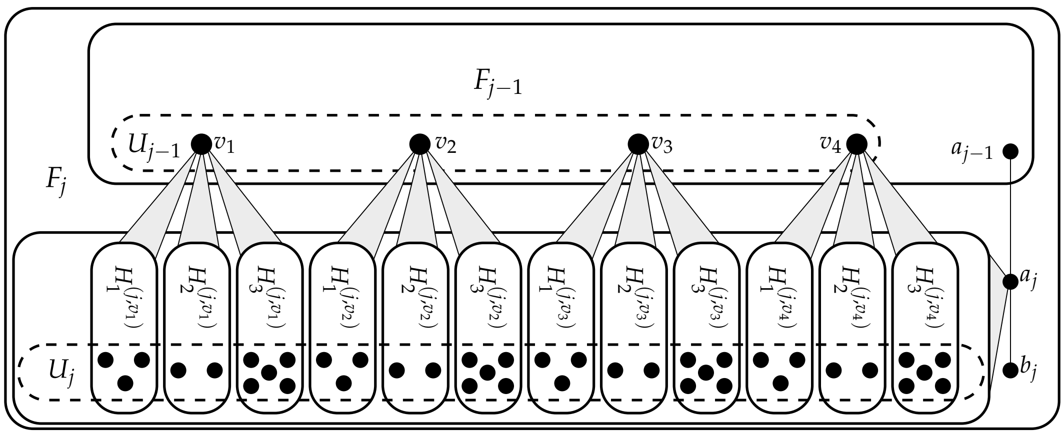

Figure 1: An example of the construction of F_j from F_{j-1} where $k = 3$.

Let G_0 be F_ℓ. Observe that the vertices $a_0, \dots, a_\ell$ induce a path, and the vertices $b_0, \dots, b_\ell$ all have degree 1. The vertices $b_0, \dots, b_\ell$ are the only vertices of degree 1, so all automorphisms of G_0 set-wise stabilize $\{b_0, \dots, b_\ell\}$ and hence set-wise stabilize $\{a_0, \dots, a_\ell\}$. Since all vertices in $\{a_0, \dots, a_\ell\}$ have distinct degrees, these vertices are point-wise stabilized by all automorphisms of G_0. Therefore, every set $V(F_j) \setminus V(F_{j-1})$ is set-wise stabilized by every automorphism of G_0. In particular, any automorphisms of G_0 must set-wise stabilize the set $V(F_0) - \{a_0, b_0\}$ which induces a copy of $H_0 - x_0$.

It remains to show that G_0 satisfies the conditions of Theorem 3 by providing a strategy for the player to respond to the adversary's challenges. Informally, in the jth round the player will delete a vertex from the jth layer (i.e. $V(F_j) \setminus V(F_{j-1})$), and this vertex will depend on the jth group, Γ_{i_j}, and the previous vertex-deletions. The previous vertex-deletion removed a copy of the vertex $x_{i_{j-1}}$ from a copy of $H_{i_{j-1}}$ (or $j = 1$, $i_0 = 0$, and x_0 was never included in G_0). To "remove" the symmetry found in this copy of $H_{i_{j-1}}$, we aim to stabilize its copy of $y_{i_{j-1}}$. We delete the vertex x_{i_j} from the copy of H_{i_j} in the neighborhood of this copy of $y_{i_{j-1}}$, which distinguishes it from all other vertices in U_{j-1} and hence the symmetry in $H_{i_{j-1}} - x_{i_{j-1}}$ is no longer available. Instead, we have removed x_{i_j} from a copy of H_{i_j}, revealing $\mathrm{Aut}(H_{i_j} - x_{i_j}) \cong \Gamma_{i_j}$. Thus, the automorphisms allowed within F_j

are exactly those in this copy of $H_{i_j} - x_{i_j}$, and all vertices in $F_\ell \setminus F_j$ have their motion determined by the action in F_j.

Back to the formal proof, we first show that we can localize our study of the automorphisms of $G_0 - X$ for certain sets of vertices X.

Claim 7. *Fix $j \in \{0, \ldots, \ell\}$ and $X = \{v_1, \ldots, v_j\}$ where $v_{j'} \in V(F_{j'}) \setminus (V(F_{j'-1}) \cup U_{j'} \cup \{a_{j'}, b_{j'}\})$ for all $j' \in \{1, \ldots, j\}$. Then $\mathrm{Aut}(G_0 - X) \cong \mathrm{Aut}(F_j - X)$.*

Proof of Claim 7: Observe that the vertices $b_0, \ldots, b_\ell$ remain the only vertices in $G_0 - X$ of degree 1, and the vertices $a_0, \ldots, a_\ell$ continue to have distinct degrees. Thus, the vertices $a_0, \ldots, a_\ell$ are point-wise stabilized by $\mathrm{Aut}(G_0 - X)$ and hence the sets $V(F_i) \setminus V(F_{i-1})$ are set-wise stabilized by $\mathrm{Aut}(G_0 - X)$. Specifically, the sets $V(F_{j'+1}) \setminus V(F_{j'})$ are set-wise stabilized by $\mathrm{Aut}(G_0 - X)$ for all $j' \in \{j, \ldots, \ell - 1\}$. This implies that every automorphism in $\mathrm{Aut}(F_{j'+1} - X)$ is also an automorphism of $\mathrm{Aut}(F_{j'} - X)$ when restricted to $V(F_{j'} - X)$. We will show that this map from $\mathrm{Aut}(F_{j'+1} - X)$ to $\mathrm{Aut}(F_{j'} - X)$ is a bijection for all $j' \in \{j, \ldots, \ell - 1\}$, implying there is natural bijection between $\mathrm{Aut}(F_\ell - X)$ and $\mathrm{Aut}(F_j - X)$.

Every vertex $u \in V(F_{j'+1} - X) \setminus V(F_{j'})$ is contained in $H_i^{(j'+1,v)}$ for some vertex $v \in U_{j'}$ and $i \in \{1, \ldots, k\}$. Since $V(H_i^{(j'+1,v)}) \cap X = \emptyset$, this subgraph $H_i^{(j'+1,v)}$ has no non-trivial automorphisms. Therefore, for every automorphism σ of $F_{j'} - X$, there is exactly one isomorphism of $F_{j'+1} - X$ that extends σ and maps $V(H_i^{(j'+1,v)})$ to $V(H_i^{(j'+1,\sigma(v))})$. Hence, the action of an automorphism on each vertex $u \in V(F_{j'+1} - X) \setminus V(F_{j'})$ is determined exactly by the action of the automorphism on the vertices within $V(F_{j'} - X)$. Hence, the restriction map from $\mathrm{Aut}(F_{j'+1} - X)$ to $\mathrm{Aut}(F_{j'} - X)$ is a bijection, proving the claim. □

When $X = \emptyset$, the automorphism group of the subgraph F_0 determines the automorphism group of $G_0 - X$. Since $F_0 - \{a_0, b_0\} \cong H_0 - x_0$, we have $\mathrm{Aut}(G_0) \cong \Gamma_0$.

For a list $\Gamma_{i_1}, \ldots, \Gamma_{i_\ell}$ of groups selected from $\{\Gamma_1, \ldots, \Gamma_k\}$, define the vertices $v_1, \ldots, v_\ell$ and $u_0, u_1, \ldots, v_\ell$ where $u_0 = y_0$ and for $j \in \{1, \ldots, \ell\}$,

$$v_j = x_{i_j}^{(j,u_{j-1})}, \quad u_j = y_{i_j}^{(j,u_{j-1})}.$$

Observe that the definition of v_j and u_j depends only on u_{j-1} and Γ_{i_j}, so this definition does not violate any of the quantifiers in the statement of Theorem 4. Thus, the vertices $v_1, \ldots, v_\ell$ are valid selections of vertex-deletions for the player in response to the adversary selecting $\Gamma_{i_1}, \ldots, \Gamma_{i_\ell}$ in order.

By induction on j, we verify that $\mathrm{Aut}(G_0 - v_1 - \cdots - v_j) \cong \Gamma_{i_j}$. We will require the stronger induction hypothesis that all automorphisms of $F_j - v_1 - \cdots - v_j$ point-wise stabilize all vertices except those in $H_{i_j}^{(j,u_{j-1})} - v_j$.

Let $j \in \{1, \ldots, \ell\}$. By Claim 7, $\mathrm{Aut}(G_0 - v_1 - \cdots - v_j) \cong \mathrm{Aut}(F_j - v_1 - \cdots - v_j)$. Since $b_0, \ldots, b_j$ are the only vertices of degree 1, and they are only adjacent to $a_0, \ldots, a_j$ (which have different degrees), the vertices $a_0, \ldots, a_j$ and $b_0, \ldots, b_j$ are point-wise stabilized by $\mathrm{Aut}(F_j - v_1 - \cdots - v_j)$. Thus, $V(F_{j-1})$ is set-wise stabilized by $\mathrm{Aut}(F_j - v_1 - \cdots - v_j)$. By induction (or that $F_0 - \{a_0, b_0\} \cong H_0 - x_0$ in the case $j = 1$), $\mathrm{Aut}(F_{j-1} - v_1 - \cdots - v_{j-1}) \cong \Gamma_{i_{j-1}}$ and all vertices in F_{j-1} are point-wise stabilized by $\mathrm{Aut}(F_{j-1} - v_1 - \cdots - v_{j-1})$ except those in $H_{i_{j-1}}^{(j-1,u_{j-2})} - v_{j-1}$ (for the case $j = 1$,

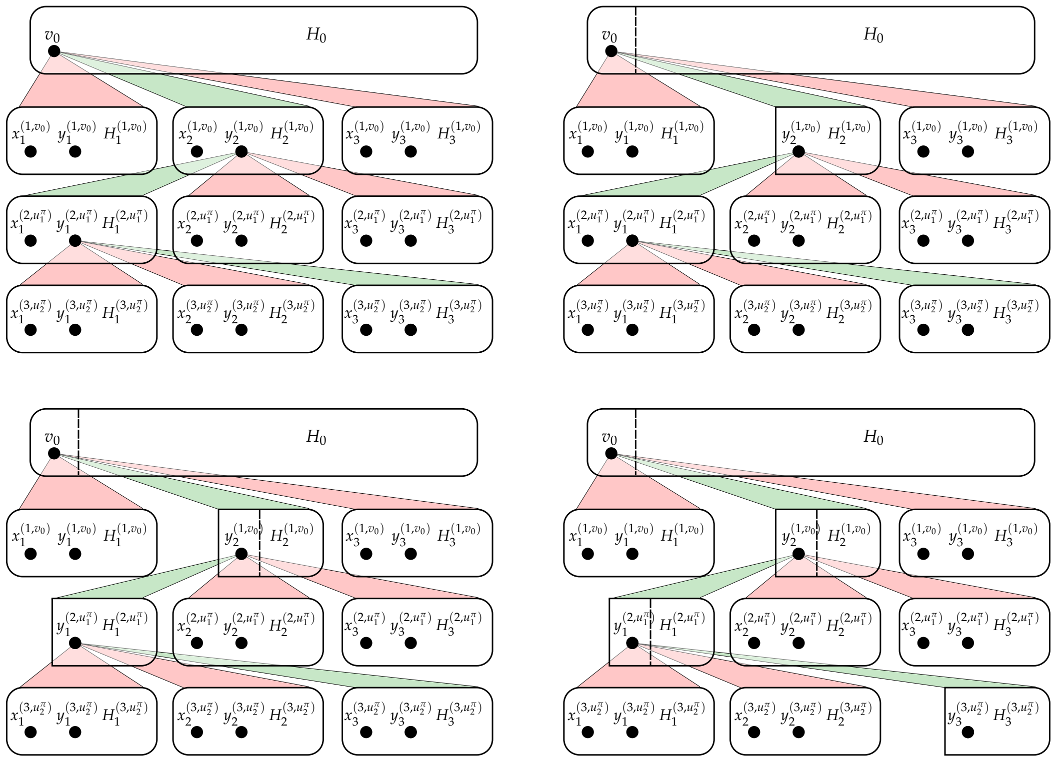

Figure 2: An example sequence of deletions with $k = 3$ where the adversary selects $\Gamma_2, \Gamma_1, \Gamma_3$.

use $H_0 - x_0$ instead of $H_{i_{j-1}}^{(j-1,u_{j-2})} - v_{j-1}$). Observe that u_{j-1} is the copy of $y_{i_{j-1}}$ in $H_{i_{j-1}}^{(j-1,u_{j-2})}$. Since deleting v_j from $F_j - v_1 - \cdots - v_j$ creates a copy of $H_{i_j} - x_{i_j}$ in the neighborhood of u_{j-1}, the vertex u_{j-1} is distinguished from the other vertices in U_{j-1}. Thus, u_{j-1} is stabilized by all automorphisms in $\mathrm{Aut}(F_j - v_1 - \cdots - v_j)$. This implies that the automorphisms in $\mathrm{Aut}(F_j - v_1 - \cdots - v_j)$ point-wise stabilize all vertices in F_{j-1}. Finally, all vertices in $V(F_j) \setminus V(F_{j-1})$ are either contained in $H_{i_j}^{(j,u_{j-1})} - v_j$ (in which case the automorphisms are given by $\mathrm{Aut}(H_{i_j} - x_{i_j})$) or are contained in a copy of H_i for some $i \in \{1, \ldots, k\}$ and H_i has no nontrivial automorphisms. Thus, all vertices of $F_j - v_1 - \cdots - v_j$ are point-wise stabilized except those in $H_{i_j}^{(j,u_{j-1})} - v_j$. Finally,

$$\mathrm{Aut}(G_0 - v_1 - \cdots - v_j) \cong \mathrm{Aut}(F_j - v_1 - \cdots - v_j) \cong \mathrm{Aut}(H_{i_j}^{(j,u_{j-1})} - v_j) \cong \Gamma_{i_j}. \qquad \square$$

The construction given in the above proof requires a large number of vertices and vertices of high degree. While the gadget given by Lemma 6 can be built using $O(|\Gamma| \log_2^2 |\Gamma| \log_2 \log_2 |\Gamma|)$ vertices[1], Babai [1] proved that for every finite group Γ there is a graph G with $\mathrm{Aut}(G) \cong \Gamma$ and $|V(G)| \leq 3|\Gamma|$. Can graphs with $O(|\Gamma|)$ vertices be used to satisfy Lemma 6? Also, the

[1] The construction of Lemma 5 from [4] has order $O(|\Gamma|^4)$, but can be replaced by a construction of Sabidussi [6] with $O(|\Gamma| \log_2 |\Gamma| \log_2 \log_2 |\Gamma|)$ vertices. Then, carefully applying the construction of Lemma 6 to Sabidussi's construction, the number of vertices is increased by a multiplicative factor of $O(\log_2 |\Gamma|)$.

constructions used here contain vertices of high degree. Does there exist a constant D so that Theorem 3 is satisfied with the maximum degree of G_0 at most D?

References

[1] L. Babai. On the minimum order of graphs with given group. *Canad. Math. Bull.*, **17**(4) (1974).

[2] J. A. Bondy. A graph reconstructor's manual. In *Surveys in combinatorics*, volume 166 of *London Math. Soc. Lecture Note Ser.*, pp 221–252. Cambridge Univ. Press, Cambridge, (1991).

[3] R. Frucht. Herstellung von Graphen mit vorgegebener abstrakter Gruppe. *Compositio Math.*, **6** (1939).

[4] S. G. Hartke, H. Kolb, J. Nishikawa, D. Stolee, Automorphism groups of a graph and a vertex-deleted subgraph. *Elec. J. Comb.*, **17** (2010), # R134, 8pp.

[5] B. D. McKay. Isomorph-free exhaustive generation. *J. Algorithms*, **26**(2), (1998).

[6] G. Sabidussi. On the minimum order of graphs with a given automorphism group. *Monatsh. Math.*, **63**, pp 124–127, (1959).