ReachFewL = ReachUL

Brady Garvin   Derrick Stolee*   Raghunath Tewari
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August 15, 2011

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- $\mathsf{NL} \stackrel{?}{=} \mathsf{UL}$ — ???, 20??

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- For each $x \in A$ there is exactly one accepting computation path from the initial configuration.

Log-space Classes and Reachability

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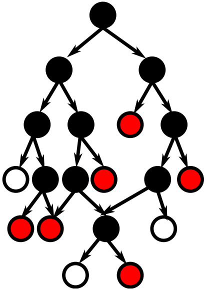

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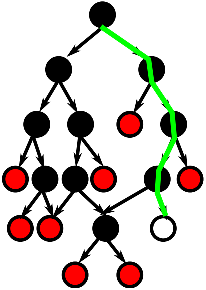

Contains:
Dir. Planar Reach
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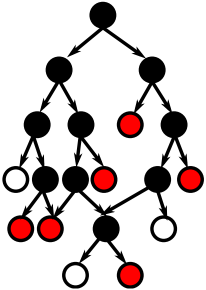

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Much ado about UL

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- $\mathsf{UL}/\mathrm{poly} = \mathsf{NL}/\mathrm{poly}$. Reinhardt, Allender, 2002

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- UL/poly $=$ NL/poly. Reinhardt, Allender, 2002

- Planar Reachability is in UL Bourke, T—, V—, 2009

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$$\mathrm{UL} \subseteq \mathrm{FewL} \subseteq \mathrm{NL}$$

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$$\mathsf{StrongUL} \subseteq \mathsf{ReachUL} \subseteq \mathsf{UL}$$

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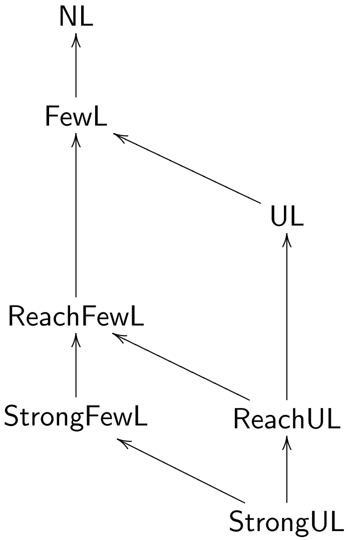
$$\text{StrongFewL} \subseteq \text{ReachFewL} \subseteq \text{FewL}$$

(ReachUL and ReachFewL originally defined by Buntrock, Jenner, Lange, and Rossmanith in 1991.)

Unambiguous Complexity Classes



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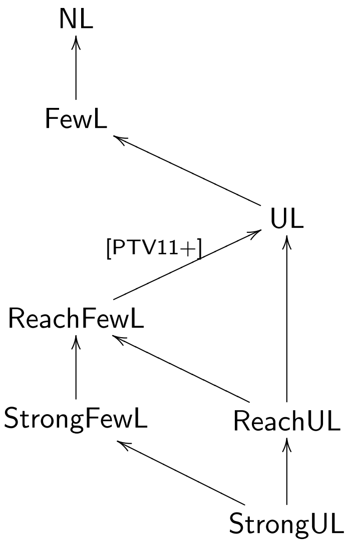

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- ReachUL $\subseteq \mathrm{SPACE}\left[\frac{\log^2(n)}{\log\log(n)}\right]$. Allender, Lange, 1998
- ReachFewL $\subseteq$ UL $\cap$ coUL Pavan, T—, V—, 2011

Complete Problems

$$L_{ru} = \{\langle G, s, t\rangle : G \text{ is a graph with exactly one path from } s \text{ to } t,$$
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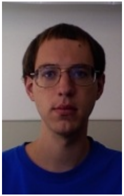
$$L_{rf}^{(k)} = \{\langle G, s, t\rangle : G \text{ is a graph with a path from } s \text{ to } t,$$
$$\text{there are at most } n^k \text{ paths from } s \text{ to any other vertex in } G\}.$$

Each language in ReachFewL reduces to $L_{rf}^{(k)}$ for some k.

Main Theorem



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$$\mathsf{ReachFewL} = \mathsf{ReachUL}$$

A Lemma About Oracles

Lemma

$\mathsf{L}^{\mathsf{ReachUL}} = \mathsf{ReachUL}$

Proof.

We can assume O is a ReachUL-complete oracle.

There are two terminal configurations: "accept" and "reject."

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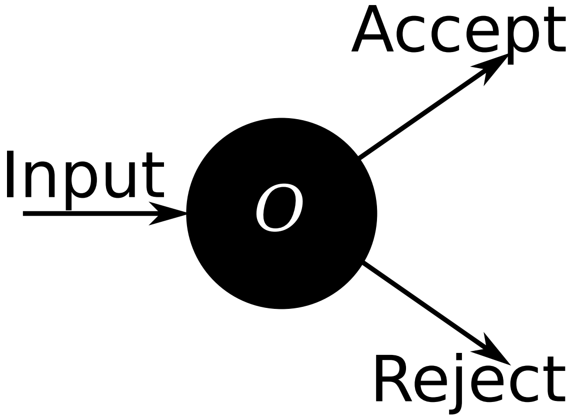

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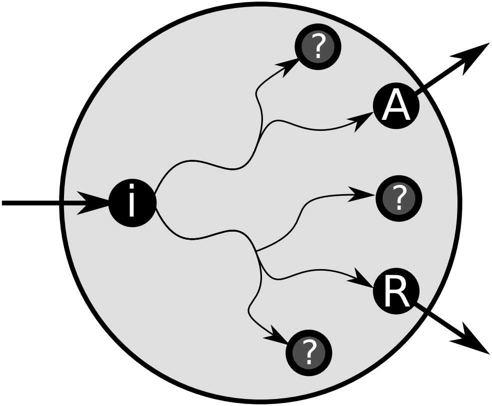

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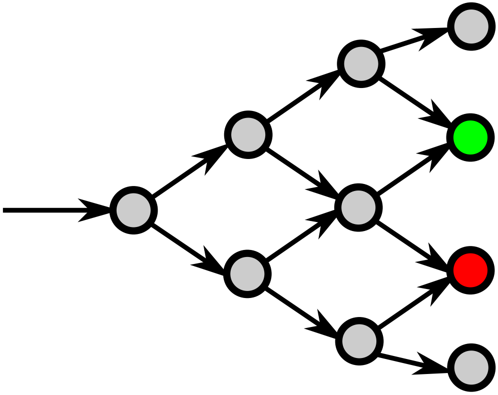

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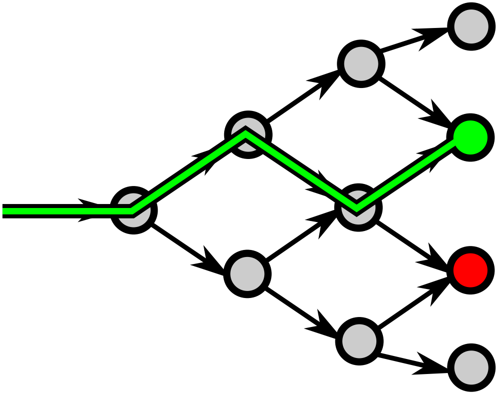


Distance Isolation

A (weighted) graph G with vertex s is *distance-isolated* with respect to s if there is no vertex t so that there are two paths from s to t of the same weight.

A Hashing Result

Theorem (Fredman, Komlós and Szemerédi, 1984)

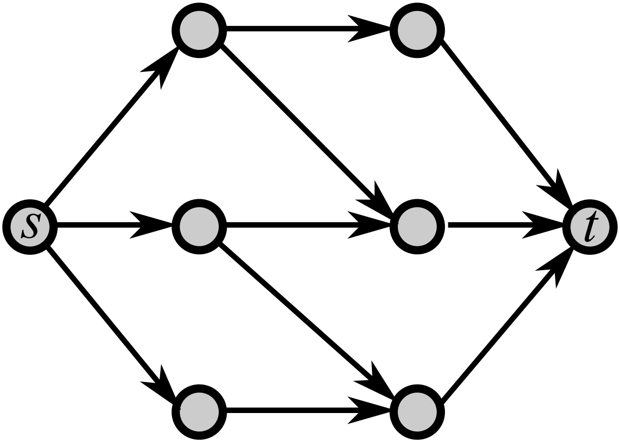
Let c be a constant and S be a set of n-bit integers with $|S| \leq n^c$. Then there is a c' and a $(c' \log n)$-bit prime number p so that for any $x \neq y \in S$, we have $x \not\equiv y \pmod{p}$.

Hashing to Distance Isolation

Lemma

Let G be a graph with edges $E(G) = \{e_1, e_2, \ldots, e_\ell\}$. If G has at most n^k paths from u to any vertex $v \in V(G)$, then there is a prime $p \leq n^{k'}$, for some constant k', such that the weight function $w_p : E(G) \to \{1, \ldots, p\}$ given by $w_p(e_i) = 2^i \pmod{p}$ defines a weighted graph G_{w_p} which is distance isolated with respect to u.

Hashing to Distance Isolation



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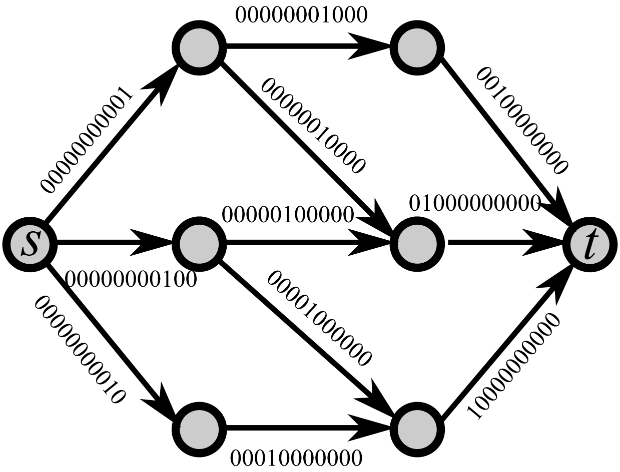

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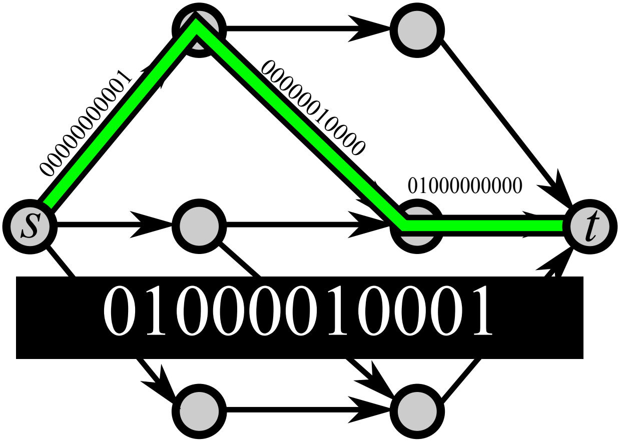

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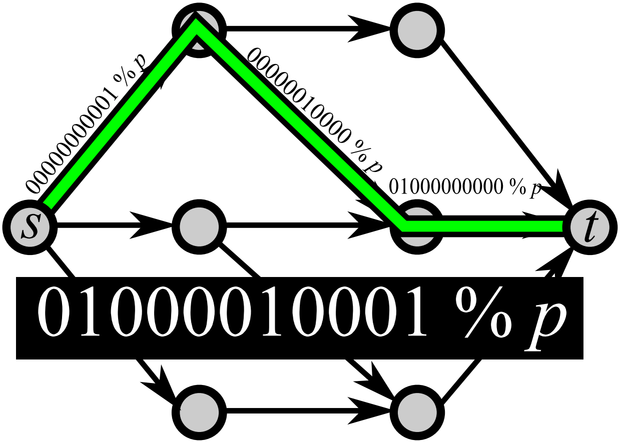

Layering Transformation

Let G be a graph. The *layered graph* $\mathrm{lay}(G)$ is the graph on vertex set $V(G) \times \{0, \ldots, n(G)\}$ with edges $(u, i) \longrightarrow (v, i + 1)$ whenever $u \longrightarrow v$ is in G.

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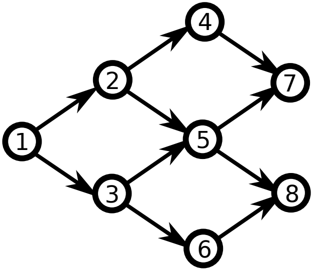

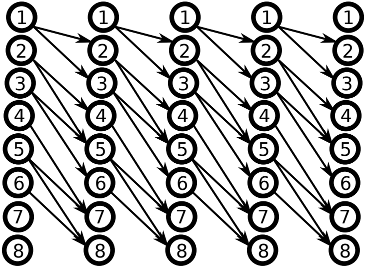

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There is a path of distance d from s to t in G if and only if there is a path from $(s, 0)$ to (t, d) in $\mathrm{lay}(G)$.

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Corollary

G is distance-isolated with respect to s if and only if $\mathrm{lay}(G)$ is reach-unique with respect to $(s, 0)$.

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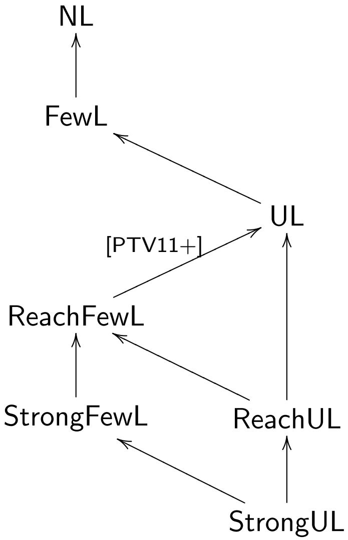
6. If all attempts fail, reject.

ReachFewSearch(G, s, t)

Require: G has at most n^k paths from s to any other vertex.
Ensure: Accepts if and only if there is a path from s to t in G.

```
for all primes p ∈ {1, ..., n^{k'}} do
    Define w_p(e_i) = 2^i (mod p).
    Construct G_{w_p}.
    Construct lay(G_{w_p}).
    if IsReachUnique(lay(G_{w_p})) then
        for each d ∈ {1, ..., n(G_{w_p})} do
            if ReachUnique(lay(G_{w_p}), (s, 0), (t, d)) then
                return True
            end if
        end for
        return False
    end if
end for
return False
```

Unambiguous Complexity Classes: Before



Unambiguous Complexity Classes: After

$$\begin{array}{c}
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\mathsf{FewL} \\
\uparrow \\
\mathsf{UL} \\
\uparrow \\
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\uparrow \\
\mathsf{StrongFewL} \\
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\end{array}$$

ReachFewL = ReachUL

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