Searching for uniquely saturated and strongly regular graphs with coupled augmentations[1]

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H-Saturated Graphs

Definition A graph G is ***H*-saturated** if

- G does not contain H as a subgraph.
- For every $e \in E(\overline{G})$, $G + e$ contains H as a subgraph.

Turán's Theorem

Theorem (Turán, 1941) Let $r \geq 3$. If G is K_{r+1}-saturated on n vertices, then G has at most $\left(1 - \frac{1}{r}\right) \frac{n^2}{2}$ edges.

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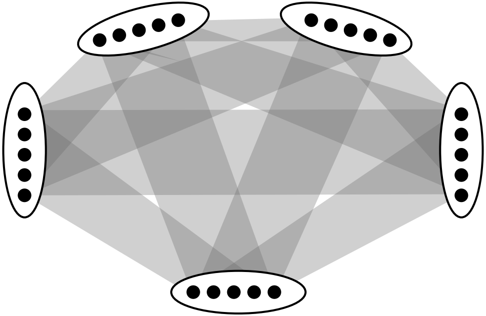

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Theorem (Erdős, Hajnal, Moon, 1964) Let $r \geq 2$. If G is K_r-saturated on n vertices, then G has at least $\binom{r-2}{2} + (r-2)(n-r+2)$ edges.

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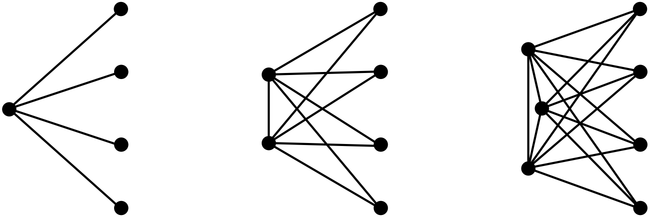

1-book 2-book 3-book

Extremal and Saturation Numbers

Definition The **extremal number** $\text{ex}(H; n)$ is the maximum number of edges in an n-vertex H-saturated graph.

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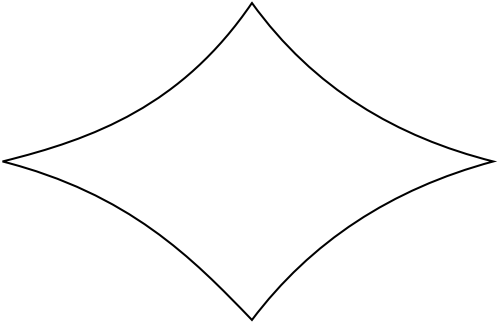

All graphs

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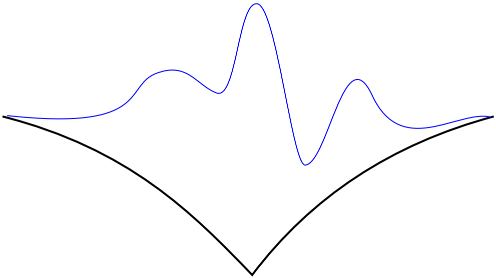

H-free graphs

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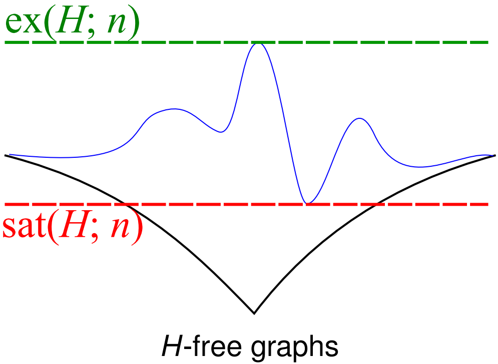

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- **Turán:**

$$\mathrm{ex}(K_{r+1}, n) \approx \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

- **Erdős, Hajnal, Moon:**

$$\mathrm{sat}(K_r; n) = \binom{r-2}{2} + (r-2)(n-r+2).$$

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Definition A graph G is **uniquely H-saturated** if G does not contain H as a subgraph and for every edge $e \in \overline{G}$ admits exactly one copy of H in $G + e$.

Uniquely C_k-Saturated Graphs

Lemma (Cooper, Lenz, LeSaulnier, Wenger, West, 2011) The uniquely C_3-saturated graphs are either **stars** or **Moore graphs** of diameter 2 and girth 5.

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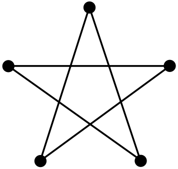
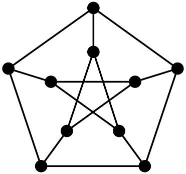
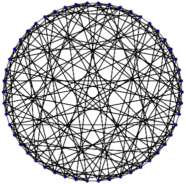


C_5



Petersen



Hoffman–Singleton

?

57-Regular Order 3250

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Theorem (Wenger, 2010) For $k \in \{6, 7, 8\}$, no uniquely C_k-saturated graph exists.

Conjecture (Wenger, 2010) For $k \geq 6$, no uniquely C_k-saturated graph exists.

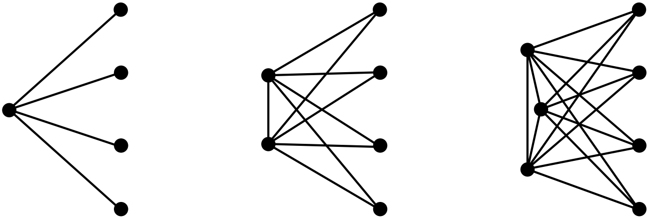
Uniquely C_k-Saturated Graphs

The method of proof is similar for uniquely C_k-saturated graphs:

- Prove that (except for a finite number of counterexamples) these graphs are **regular**.
- Develop constraints on **powers of adjacency matrix** using unique saturation.
- Prove the **eigenvalues** satisfy certain polynomial equations.
- Due to **integrality**, there are a finite set of possible matrices.

Uniquely K_r-Saturated Graphs

Let's consider $H = K_r$.



1-book 2-book 3-book

The $(r-2)$-books are uniquely K_r-saturated.

Dominating Vertices

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2**-primitive** graphs are **empty graphs**.

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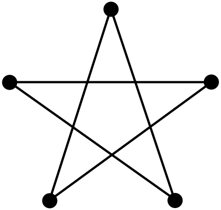
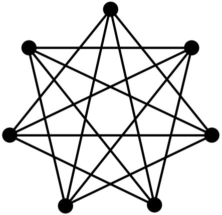
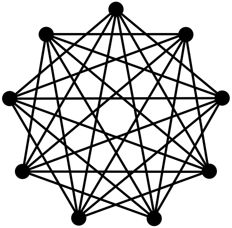


$\overline{C_5}$



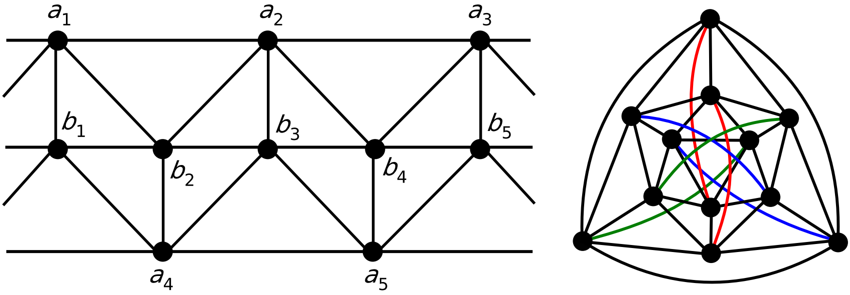
$\overline{C_7}$



$\overline{C_9}$

For $r \geq 1$, $\overline{C_{2r-1}}$ is r-primitive.

Uniquely K_4-Saturated Graphs



Previously known 4-primitive graphs (Cooper, unpublished)

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2. Is every r-primitive graph **regular**?

NO! Exists an irregular 5-primitive graph on 16 vertices!

Variables

Consider searching for uniquely K_r-saturated graphs on vertex set $\{v_1, \dots, v_n\}$.

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Use variables $x_{i,j} \in \{0, 1, *\}$ where

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- $x_{i,j} = 1$ fixes $v_i v_j \in E(G)$.
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- $x_{i,j} = *$ is **unassigned**.

If $x_{i,j} = *$ for <u>some</u> i, j, the vector $\mathbf{x}$ is a **partial assignment**.

If $x_{i,j} = *$ for <u>all</u> i, j, the vector $\mathbf{x}$ is the **empty assignment**.

Symmetries of the System

The constraints

- There is no r-clique in G.
- Every non-edge e of G has exactly one r-clique in $G + e$.

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Value-preserving permutations reflect the automorphisms of a partial assignment.

Orbital Branching

Generalizes **branch-and-bound** strategy.

Introduced by Ostrowski, Linderoth, Rossi, and Smriglio (2007) for **symmetric** optimization problems such as covering and packing.

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B2: Assign $x_{i,j} = \overline{a}$ for all $x_{i,j} \in \mathcal{O}$.

K_r-Completions

In addition to the usual constraints, we guarantee:

$x_{i,j} = 0$ **if and only if** there exists a set $S \subset [n]$ so that $x_{i,a} = x_{j,a} = x_{a,b} = 1$ for all $a, b \in S$.

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i.e. for every non-edge we add, we add a **K_r-completion**.

Also, we set $x_{i,j} = 0$ if it <u>has</u> a K_r-completion.

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SB: For every orbit $\mathcal{A}$ of $(r-2)$-subsets, select a representative $S \in \mathcal{A}$ and assign $x_{i,a} = x_{j,a} = x_{a,b} = 1$ for all $a, b \in S$.

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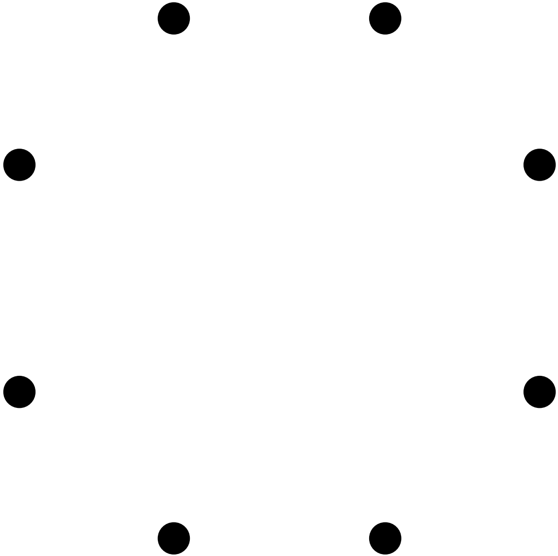
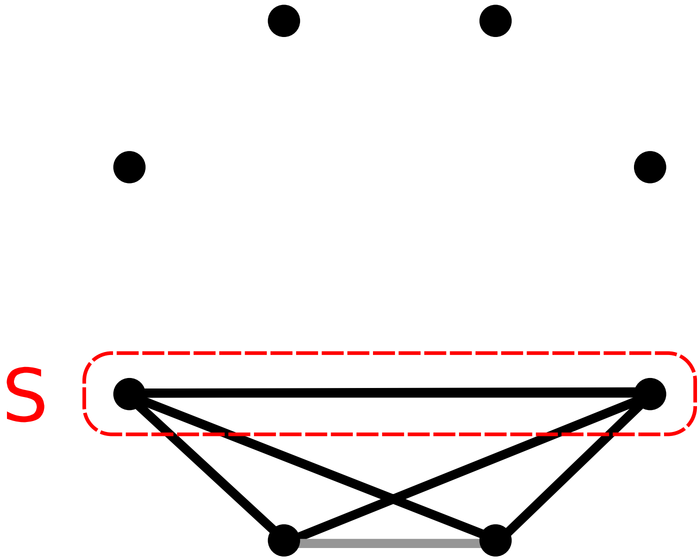
B2: Set $x_{i,j} = 1$ for all $x_{i,j} \in \mathcal{O}$.

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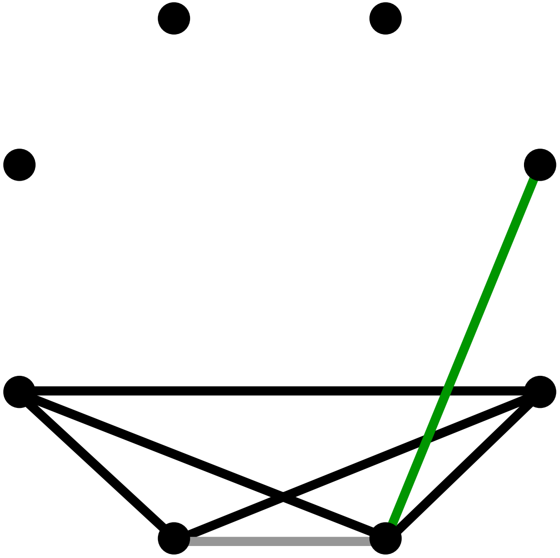

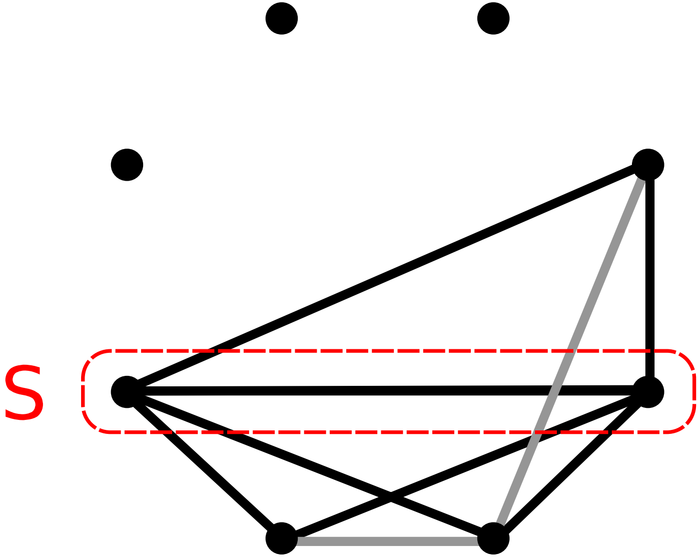
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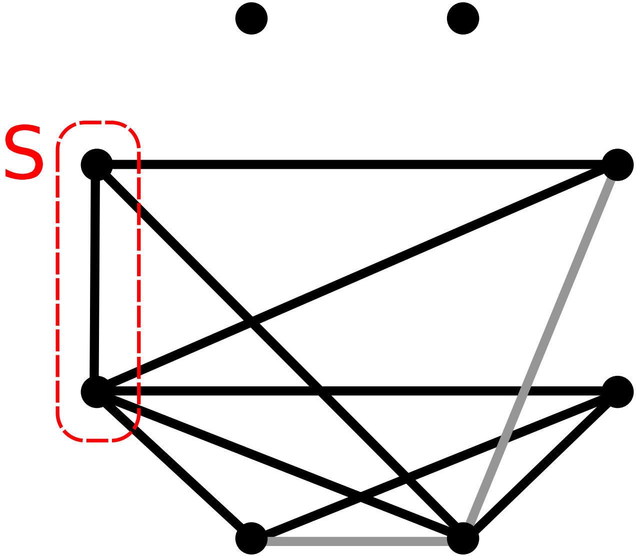
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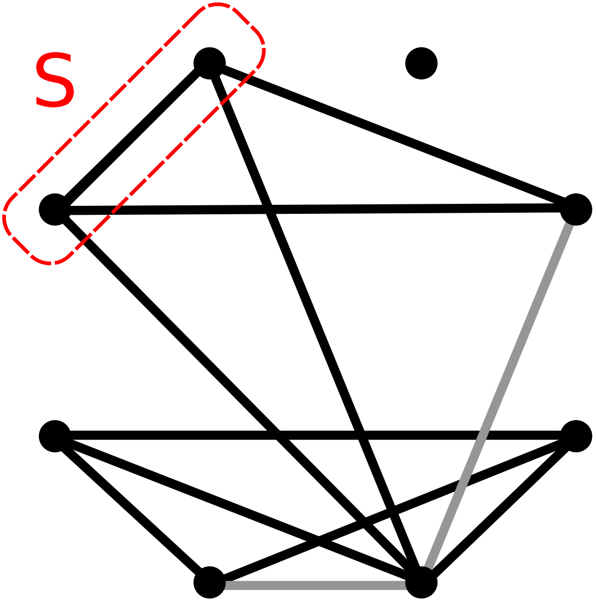
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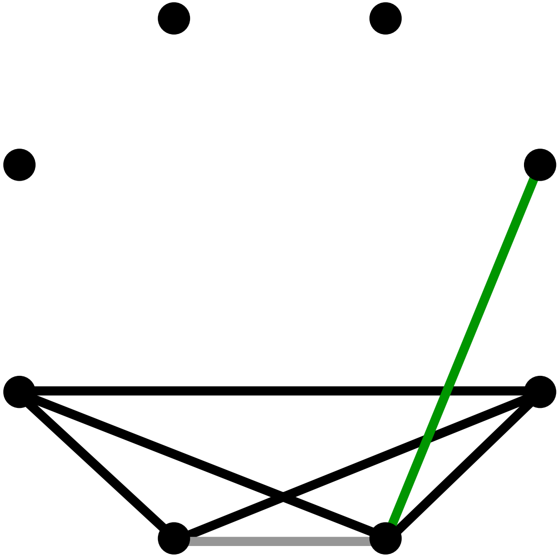

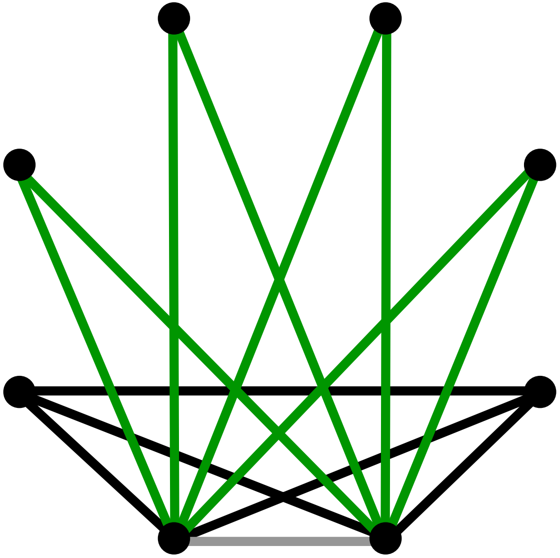

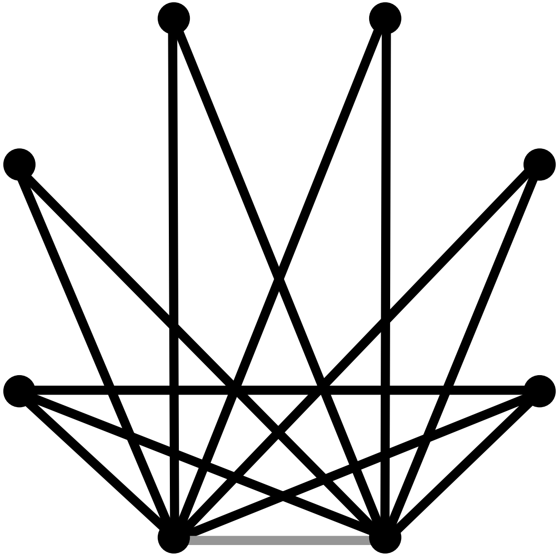

The Search Tree

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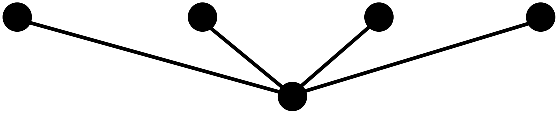

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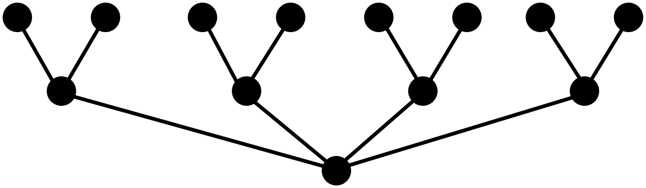

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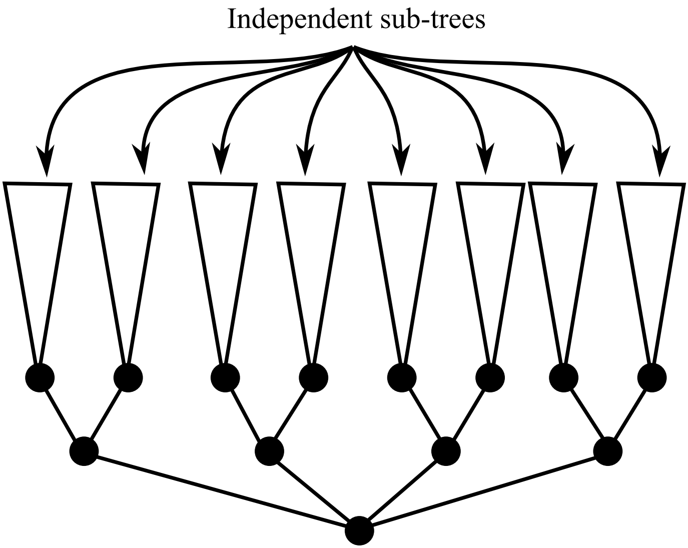

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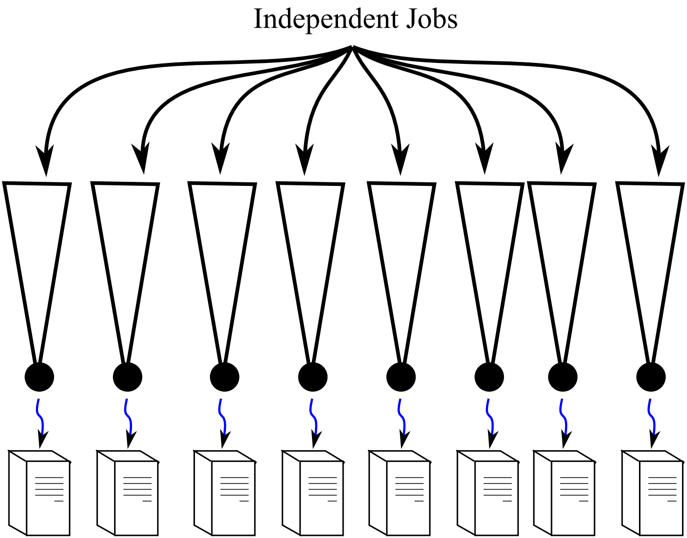

Implementation

Implemented in the **TreeSearch** library for parallelization in the Condor scheduler.

Executed on the **Open Science Grid**, a collection of supercomputers around the country.

Results

n	r	Graph List
5	3	$\overline{C_5}$
7	4	$\overline{C_7}$
9	5	$\overline{C_9}$
10	3	Petersen
10	4	M_{10}
11	6	$\overline{C_{11}}$
12	4	G_{12}
13	4	G_{13}, Payley(13)

n	r	Graph List
13	7	$\overline{C_{13}}$
15	6	$G_{15}^{(A)}, G_{15}^{(B)}$
15	8	$\overline{C_{15}}$
16	5	$G_{16}^{(A)}, G_{16}^{(B)}$
16	6	$G_{16}^{(C)}$
17	7	$\overline{C}(\mathbb{Z}_{17}, \{1,4\})$
17	9	$\overline{C_{17}}$
18	4	$G_{18}^{(A)}, G_{18}^{(B)}$

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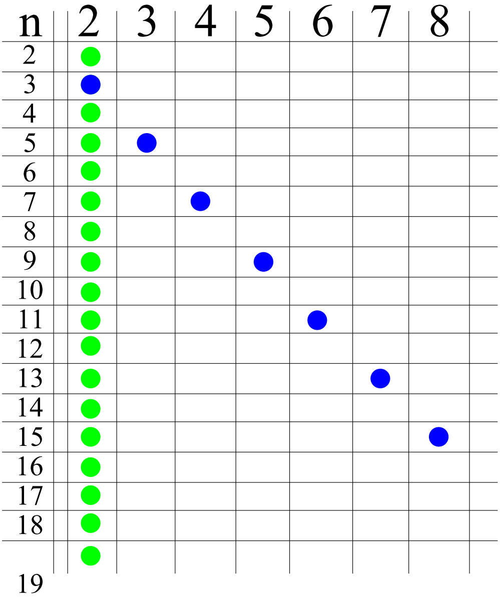
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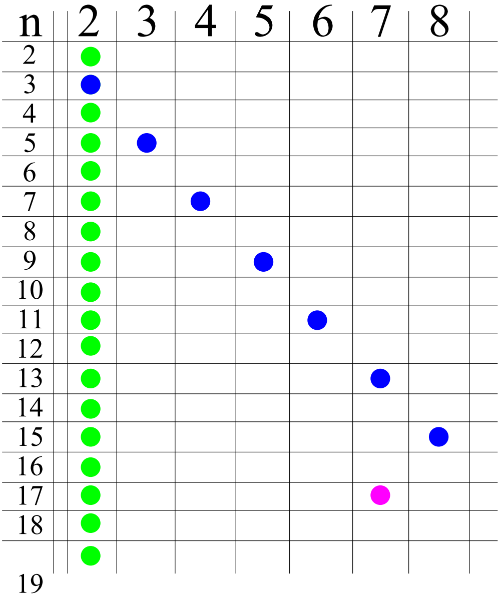
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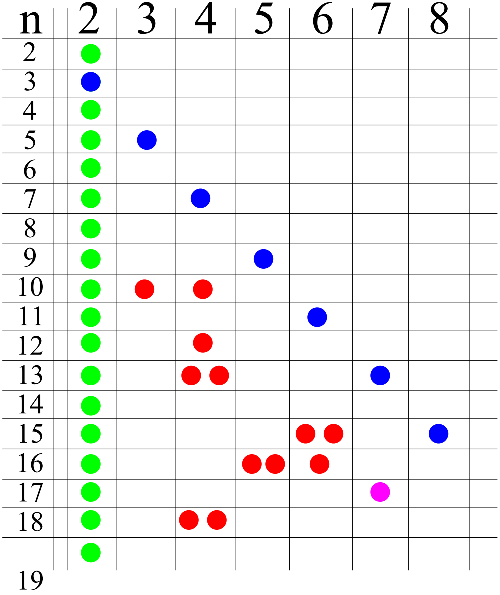
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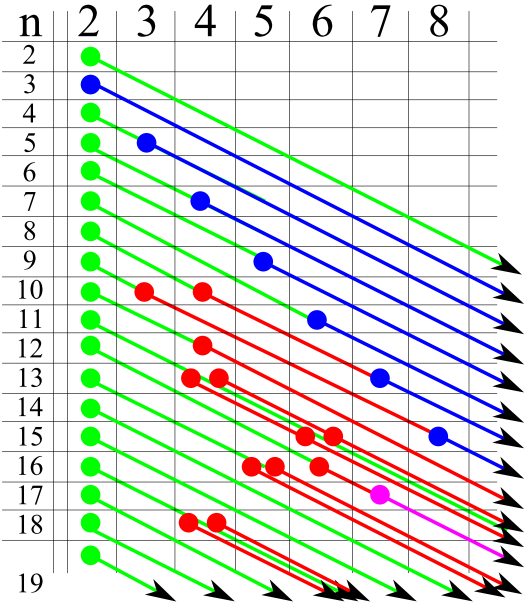
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Exhaustive Search Times

n	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$
10	0.10 s	0.37 s	0.13 s	0.01 s	0.01 s
11	0.68 s	5.25 s	1.91 s	0.28 s	0.09 s
12	4.58 s	1.60 m	25.39 s	1.97 s	1.12 s
13	34.66 s	34.54 m	6.53 m	59.94 s	20.03 s
14	4.93 m	10.39 h	5.13 h	20.66 m	2.71 m
15	40.59 m	23.49 d	10.08 d	12.28 h	1.22 h
16	6.34 h	1.58 y	1.74 y	34.53 d	1.88 d
17	3.44 d			8.76 y	115.69 d
18	53.01 d				
19	2.01 y				
20	45.11 y				

Total CPU times using Open Science Grid.

Strongly Regular Graphs

Custom Augmentations

An (n, k, λ, μ) **strongly regular graph** is a k-regular graph G on n vertices where every vertex pair $u, v \in V(G)$ has

- If uv is an edge, $|N(u) \cap N(v)| = \lambda$.
- If uv is not an edge, $|N(u) \cap N(v)| = \mu$.

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We use the λ and μ constraints for custom augmentations.

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Custom Augmentations

λ-Augmentation

Strongly Regular Graphs

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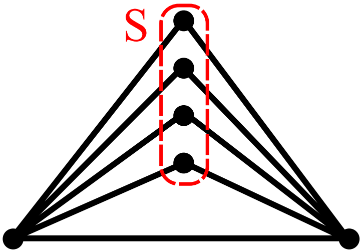

λ-Augmentation

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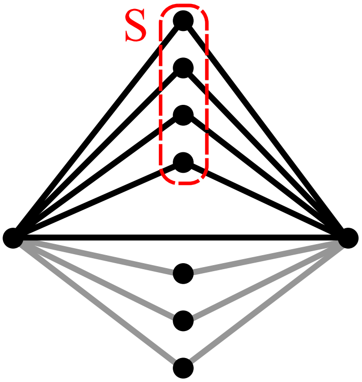

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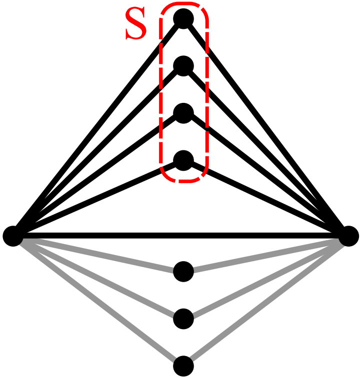

λ-Augmentation

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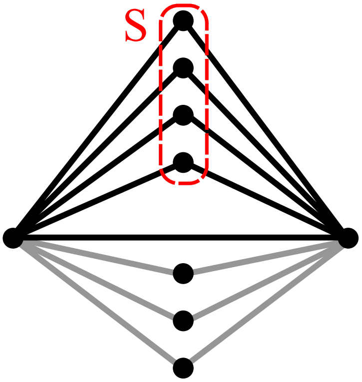
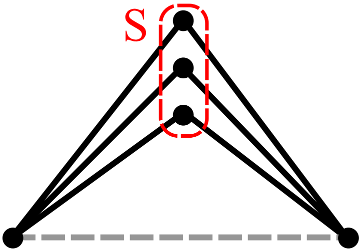


λ-Augmentation



μ-Augmentation

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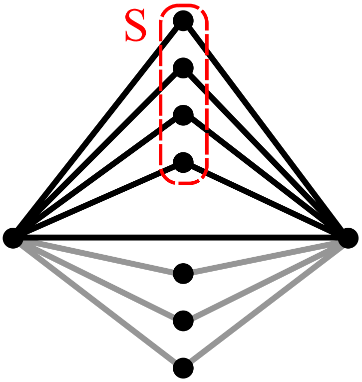
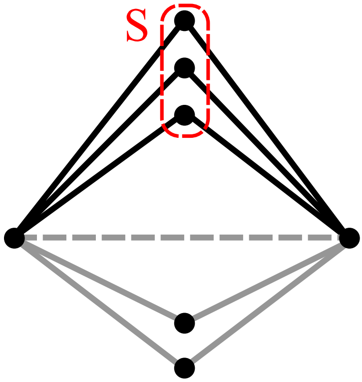


λ-Augmentation



μ-Augmentation

Strongly Regular Graphs

Still a work in progress!

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Working on interactions with LP relaxation.

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Working on interactions with LP relaxation.

Using **standard orbital branching**, we found

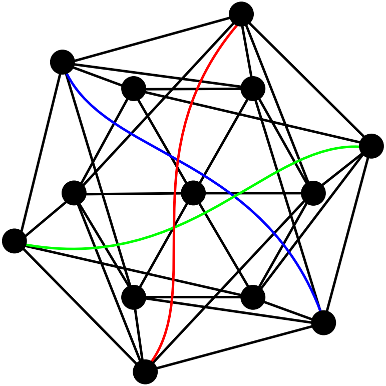
- There does not exist a $(28, 6, 3, 2, 1)$ directed strongly regular graph.
- There are at least 15 non-isomorphic $(28, 7, 2, 1, 2)$ directed strongly regular graphs.

Back to r-Primitive Graphs

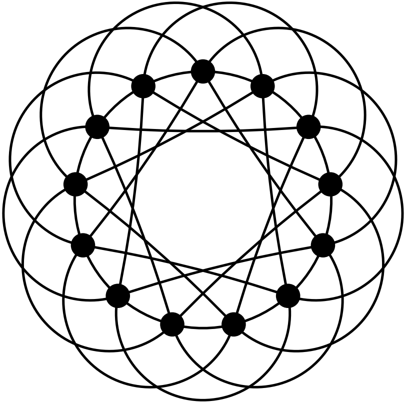
Let's get back to uniquely K_r-saturated graphs.

4-Primitive Graphs

$n = 13$



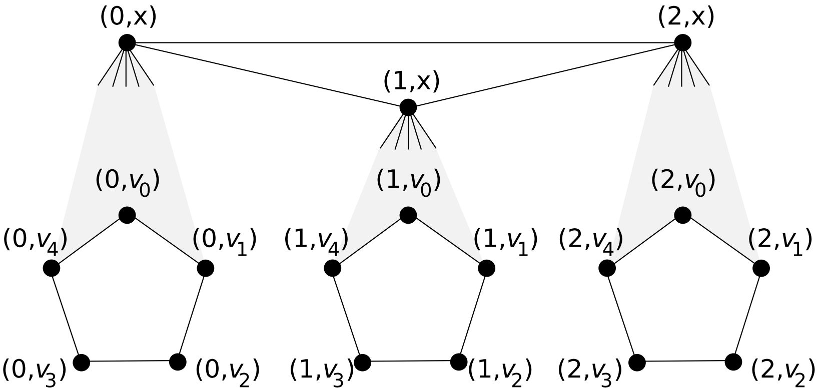
$G_{13}^{(A)}$



Payley(13)

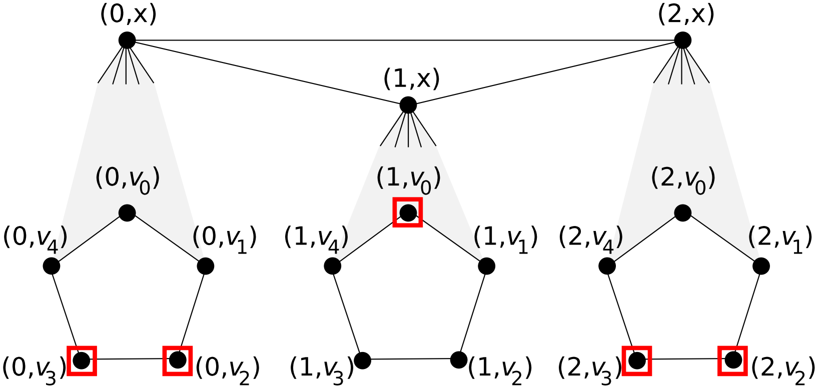
4-Primitive Graphs

$n = 18 : G^{(A)}_{18}$



4-Primitive Graphs

$n = 18 : G_{18}^{(A)}$



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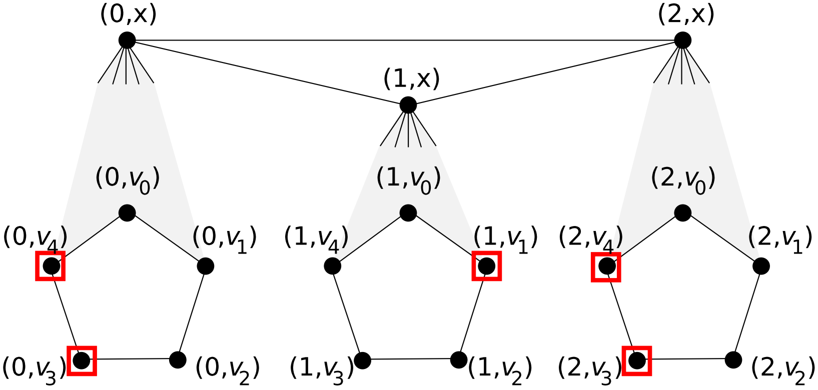

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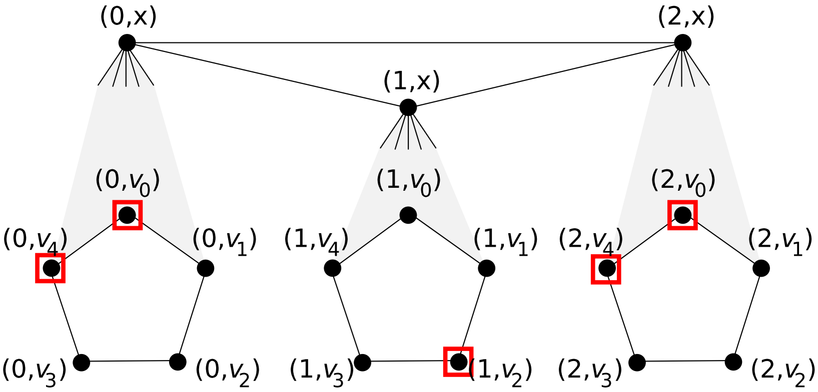

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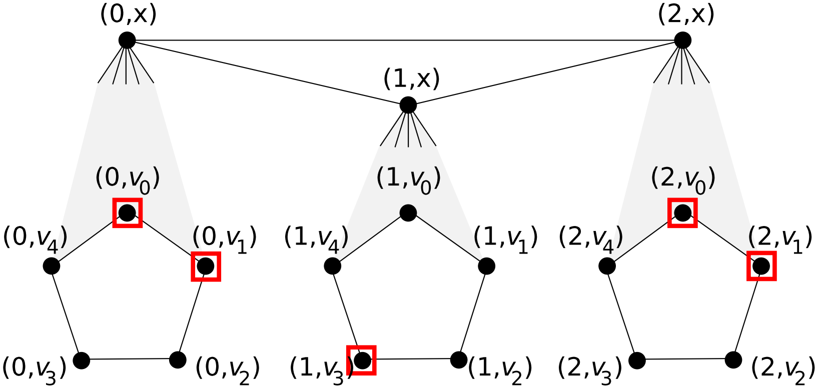

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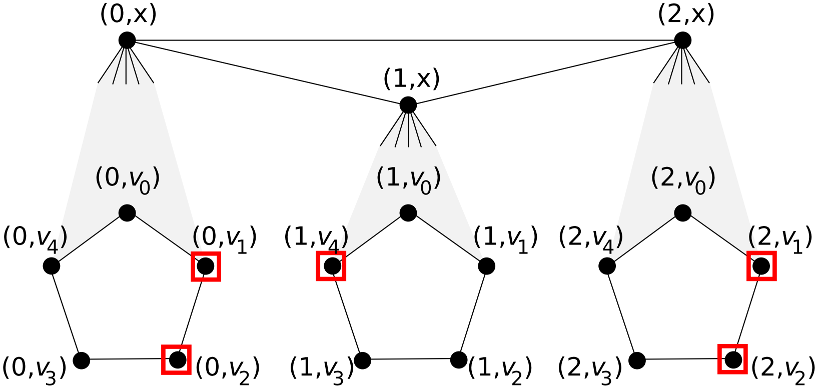

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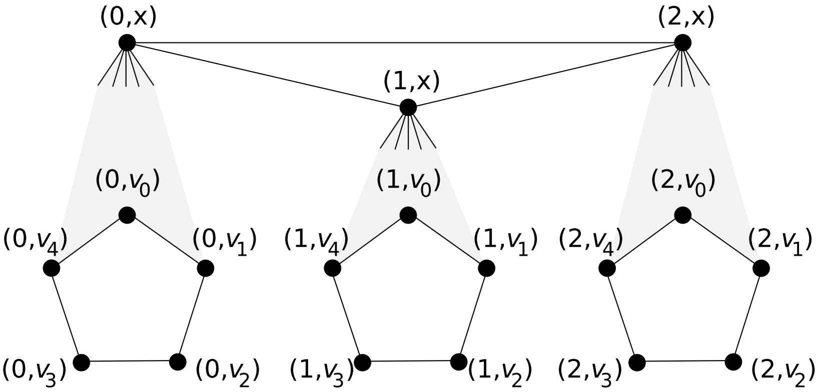
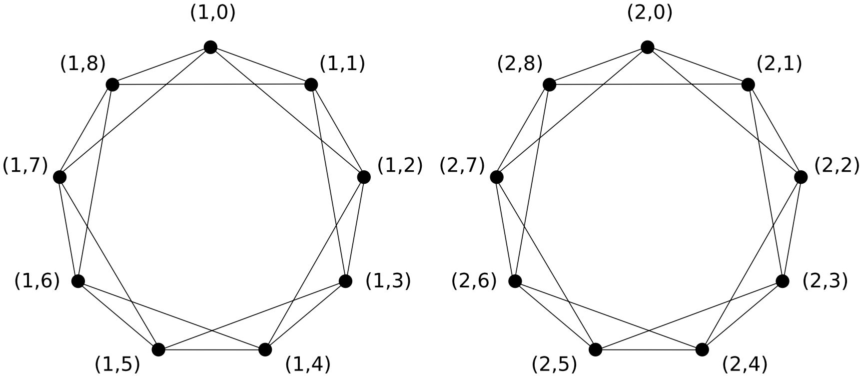


4-Primitive Graphs

$n = 18 : G_{18}^{(B)}$



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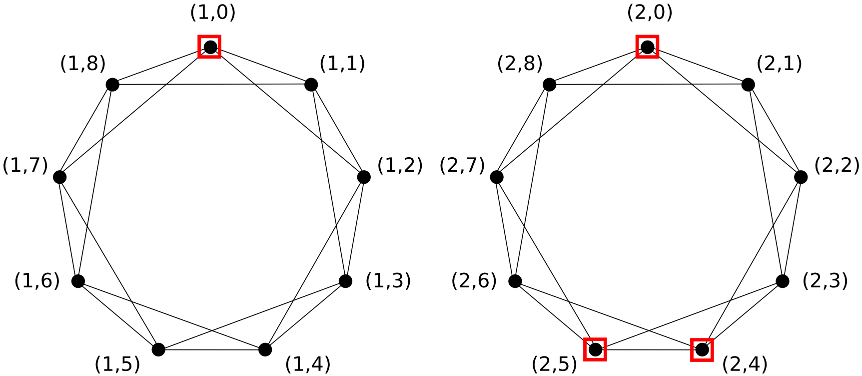

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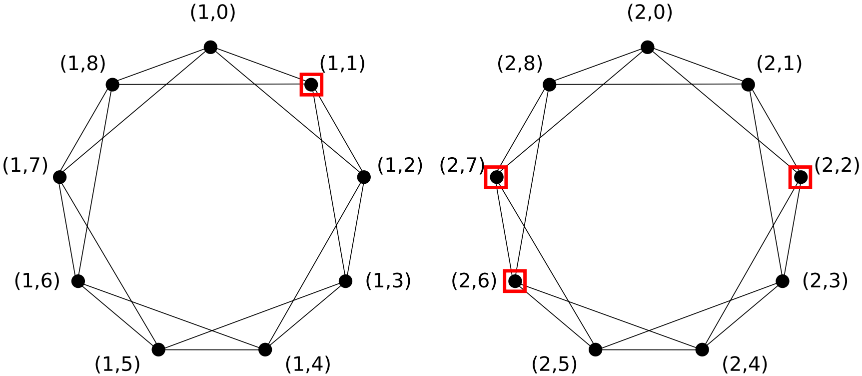

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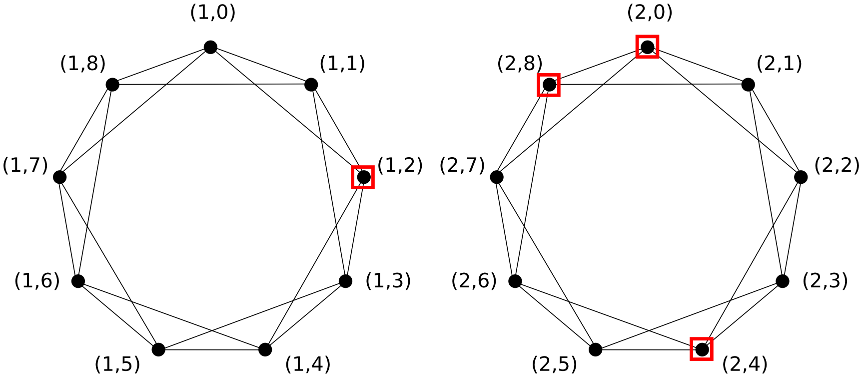

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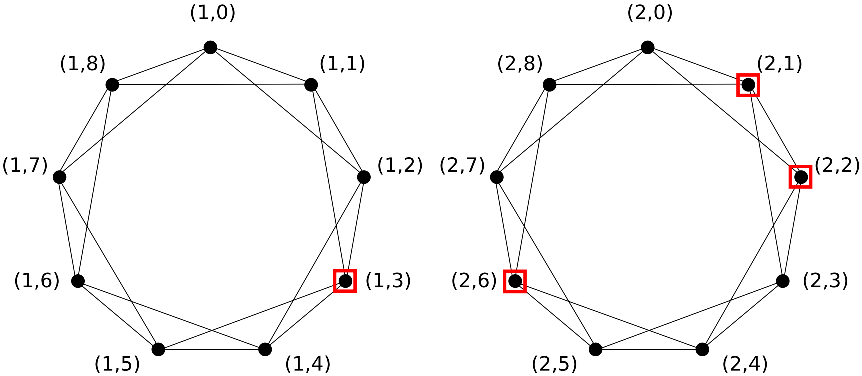

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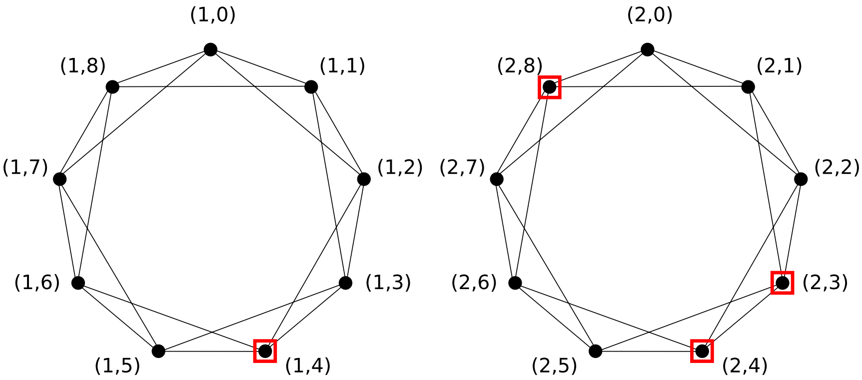

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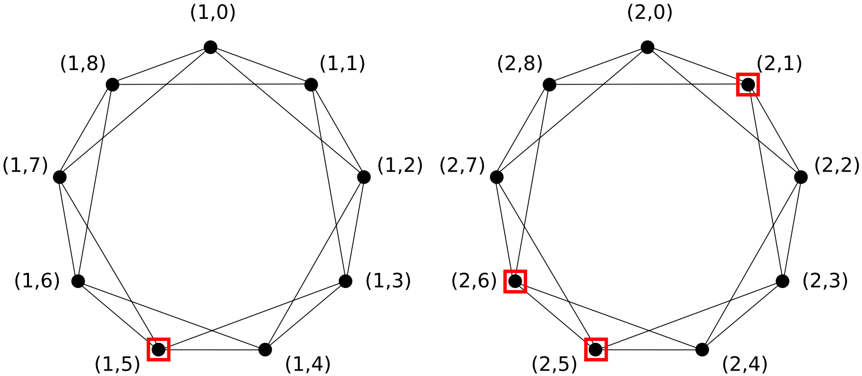

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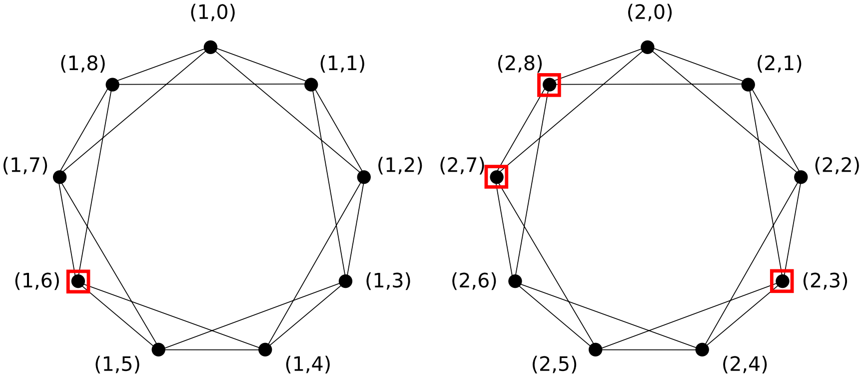

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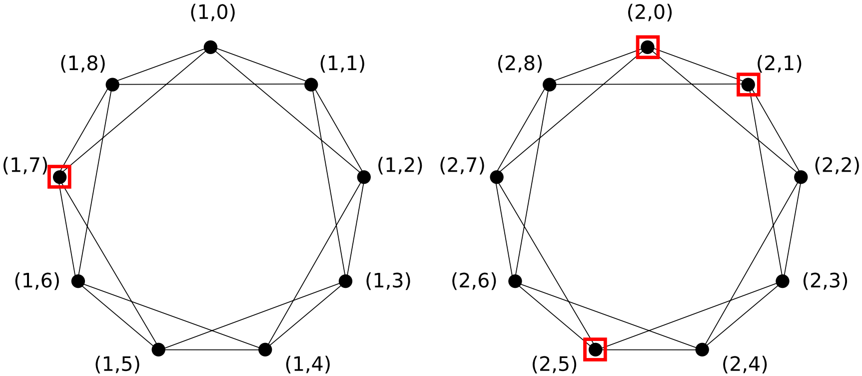

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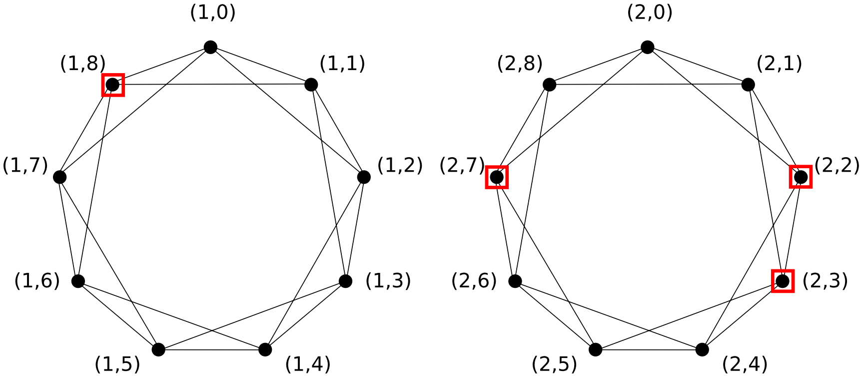

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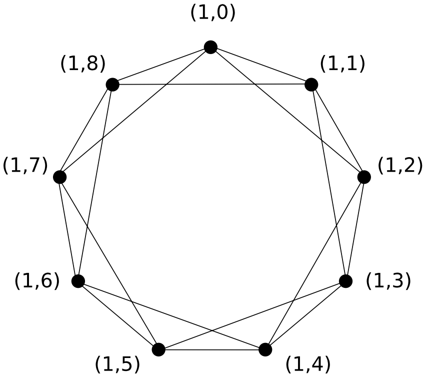
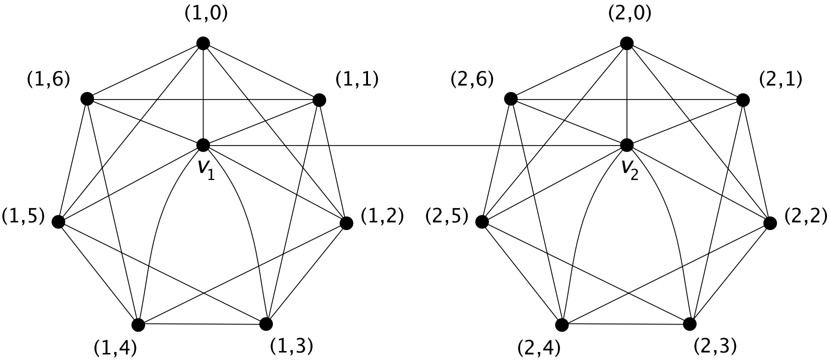


5-Primitive Graphs

$n = 16$: $G_{16}^{(A)}$ is irregular!



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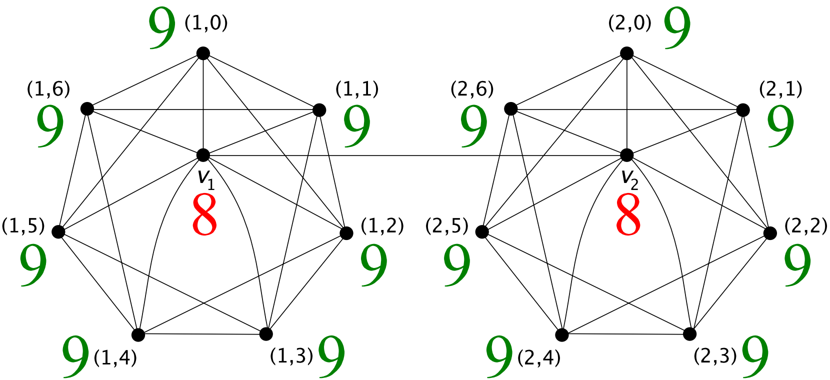

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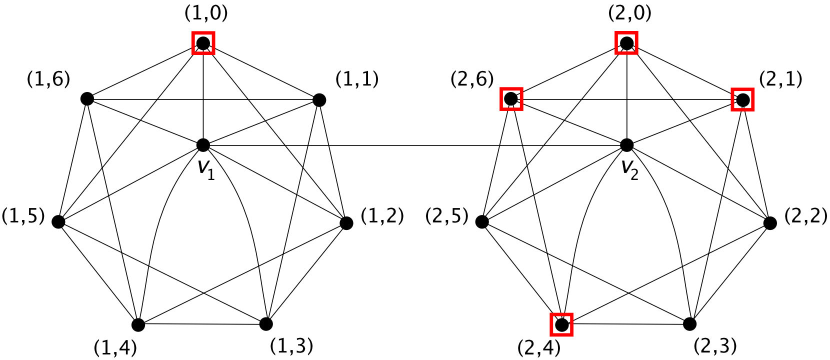

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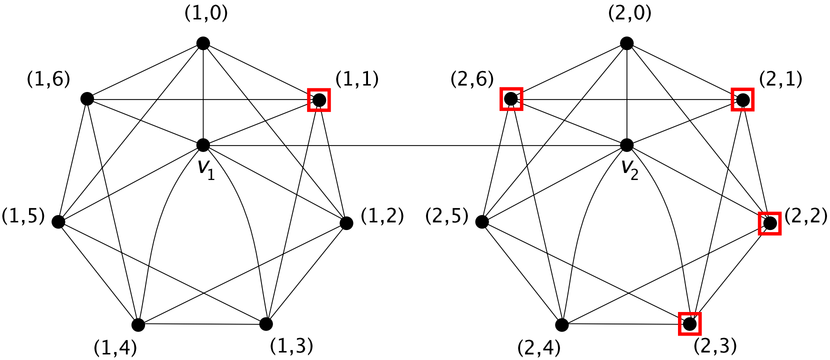

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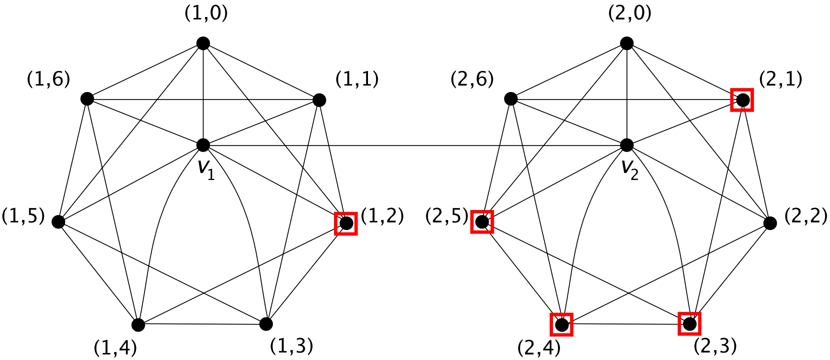

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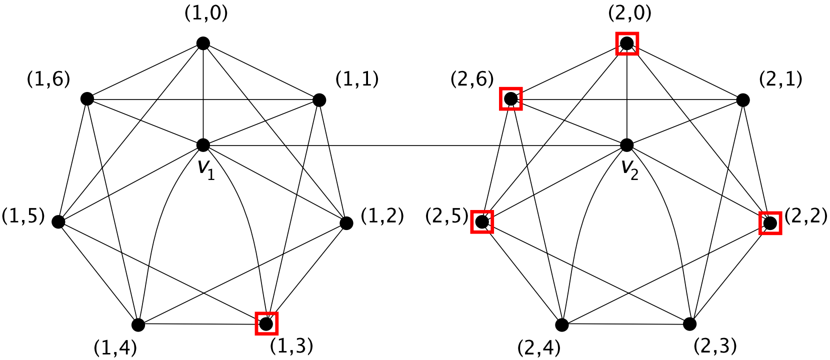

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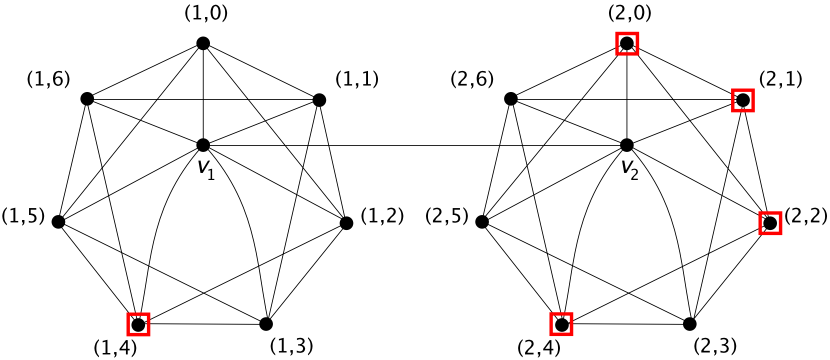

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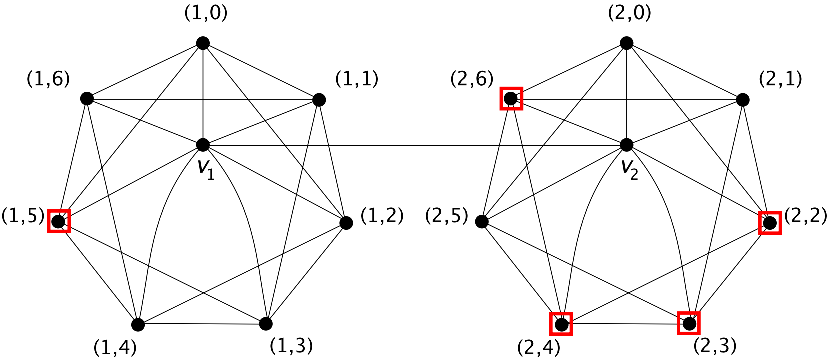

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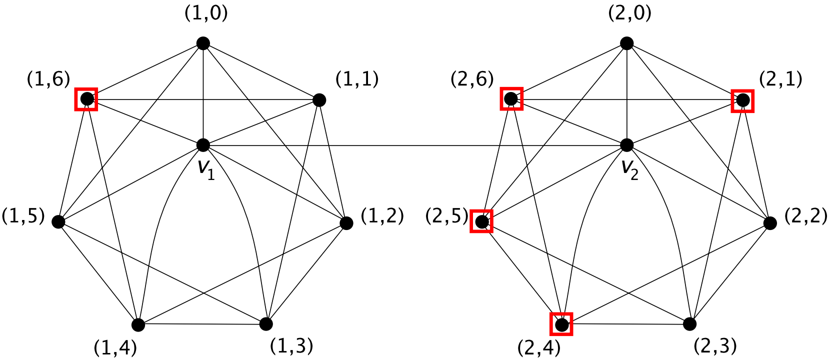

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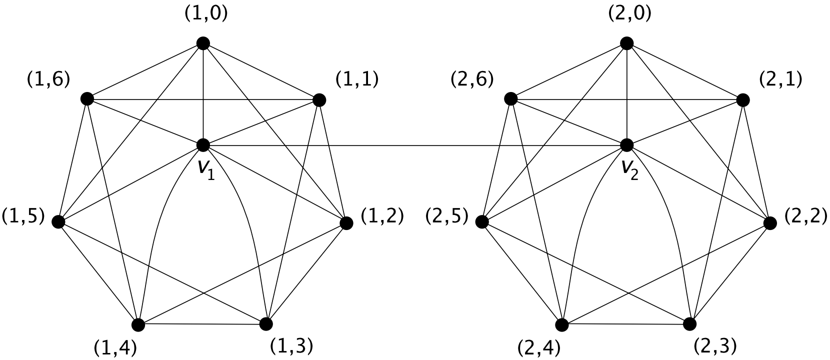

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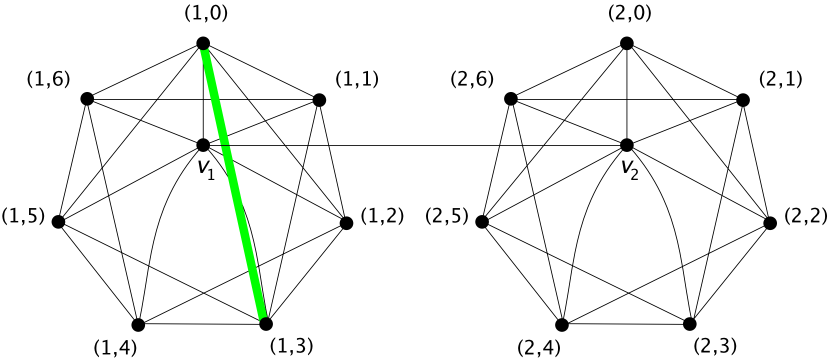

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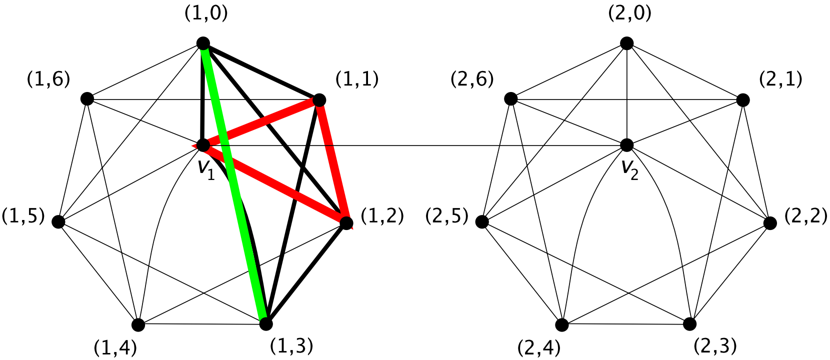

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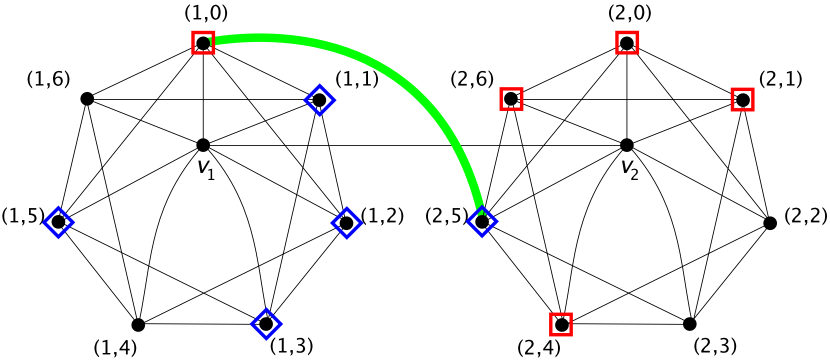

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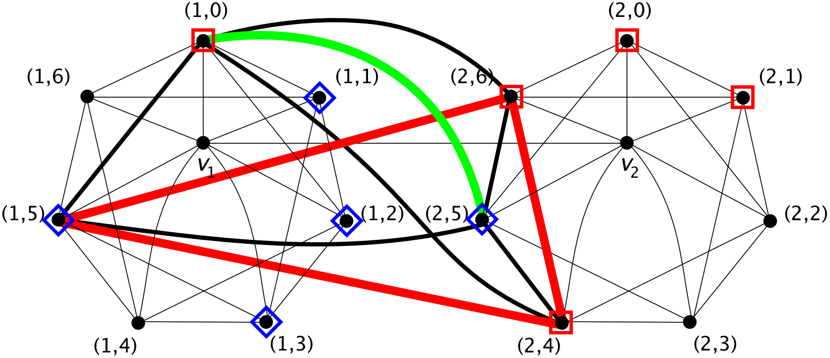

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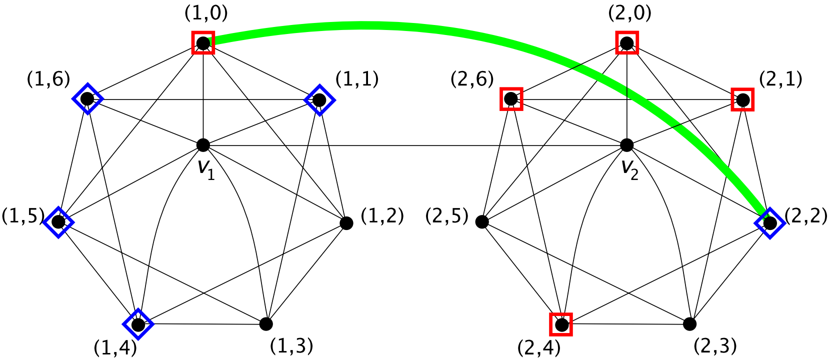

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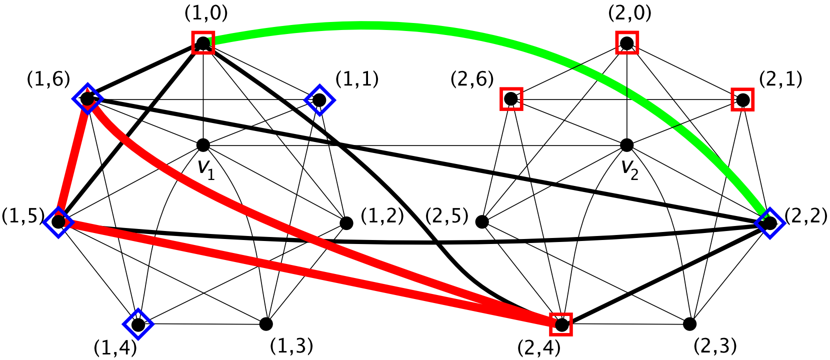

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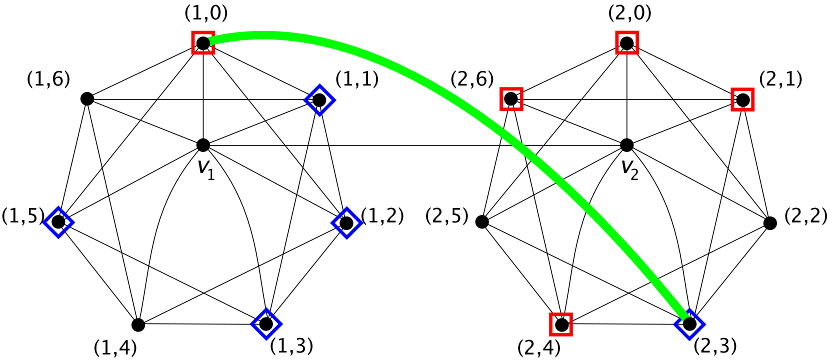

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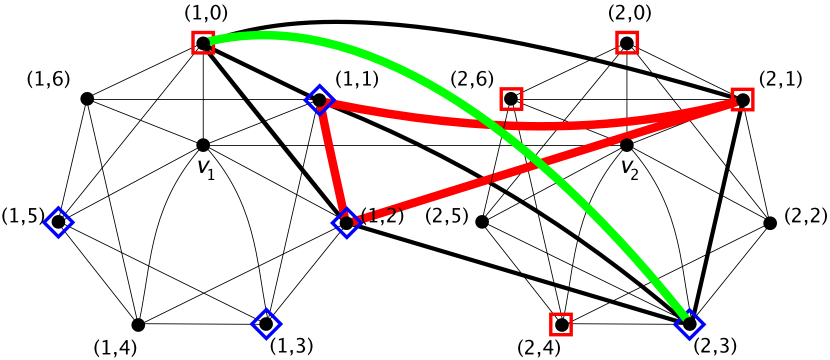
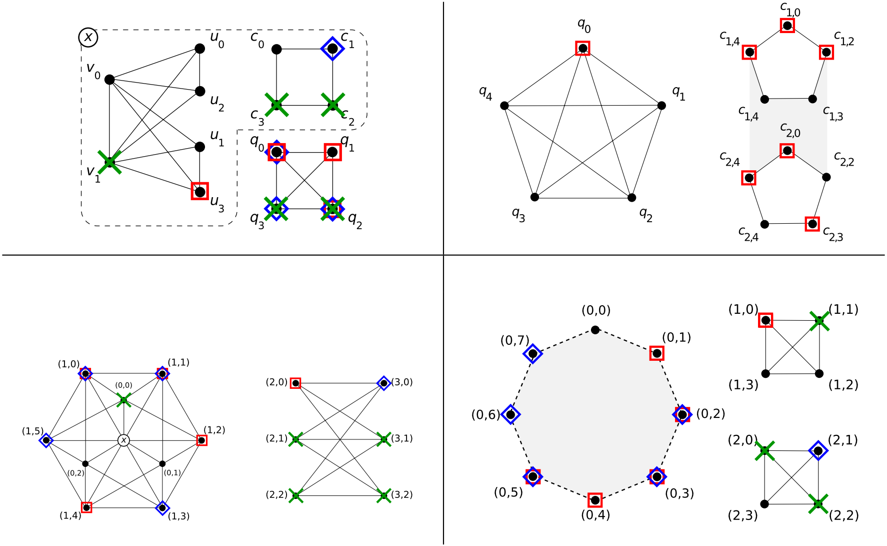
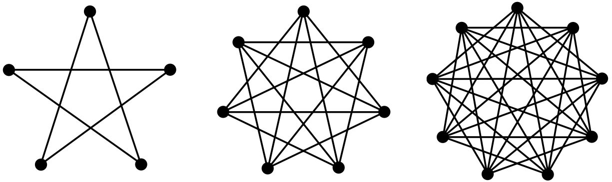


Other r-Primitive Graphs



Infinite Families

Recall: For $r \geq 1$, $\overline{C_{2r-1}}$ is r-primitive.



$\overline{C_5}$ $\overline{C_7}$ $\overline{C_9}$

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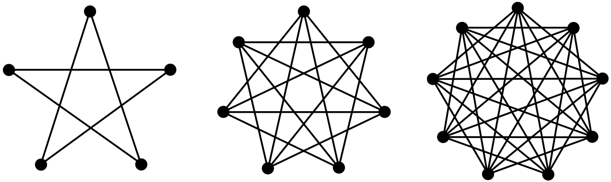

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Let n be an integer and $S \subseteq \mathbb{Z}_n$. The **Cayley complement** $\overline{C}(\mathbb{Z}_n, S)$ is the complement of the Cayley graph for $\mathbb{Z}_n$ with generator set S.

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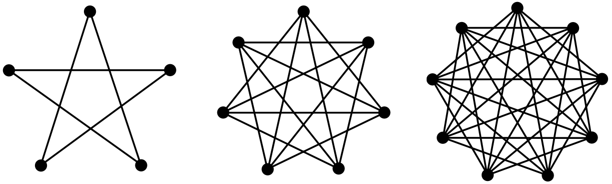

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$\overline{C}(\mathbb{Z}_{2r-1}, \{1\}) \cong \overline{C_{2r-1}}$ is r-primitive.

Two Generators

Theorem (Hartke, S—) Let $t \geq 1$, $n = 4t^2 + 1$, and $r = 2t^2 - t + 1$. The Cayley complement $\overline{C}(\mathbb{Z}_n, \{1, 2t\})$ is r-primitive.

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For $t = 1$, $r = 2$, and $\overline{C}(\mathbb{Z}_n, \{1, 2\}) \cong \overline{K_5}$.

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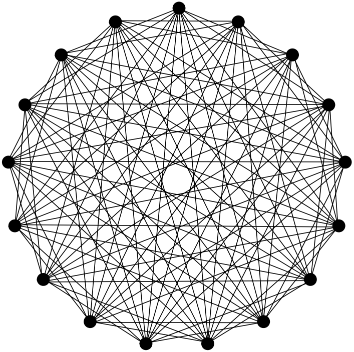
$t = 2, n = 17, r = 7$



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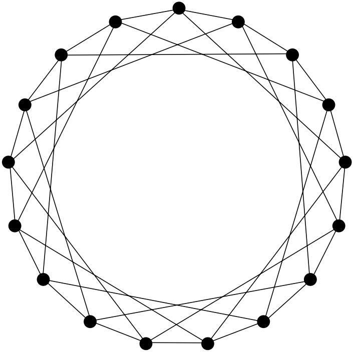

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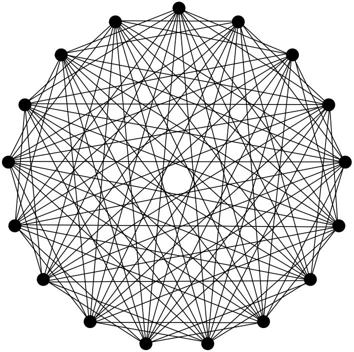

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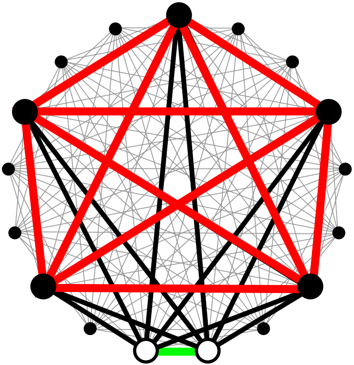

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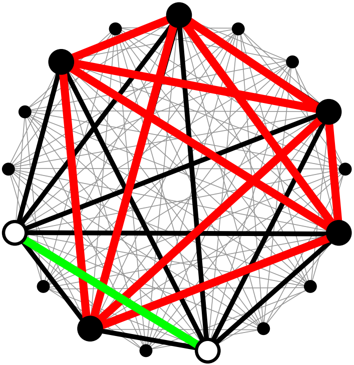

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Conjecture (Hartke, S—) Let $S \subseteq \mathbb{Z}_n$ have $|S| = 2$. The Cayley complement $\overline{C}(\mathbb{Z}_n, S)$ is r-primitive if and only if $\exists t \geq 1, n = 4t^2 + 1, r = 2t^2 - t + 1$, and $\overline{C}(\mathbb{Z}_n, S) \cong \overline{C}(\mathbb{Z}_n, \{1, 2t\})$.

Two Generators

Theorem (Hartke, S—) Let $t \geq 1$, $n = 4t^2 + 1$, and $r = 2t^2 - t + 1$. The Cayley complement $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$ is r-primitive.

t	S	r	n
1	$\{1,2\}$	2	5
2	$\{1,4\}$	7	17
3	$\{1,6\}$	16	37
4	$\{1,8\}$	29	65
5	$\{1,10\}$	46	101
6	$\{1,12\}$	67	145
7	$\{1,14\}$	92	197
8	$\{1,16\}$	121	257
9	$\{1,18\}$	154	325
10	$\{1,20\}$	191	401

Two Generators (Proof)

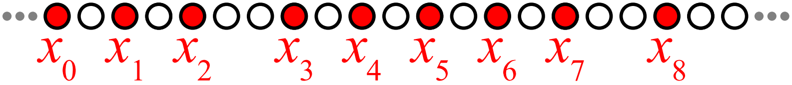
$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

Suppose $X \subseteq \mathbb{Z}_n$ is an r-clique in G.

Two Generators (Proof)

$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

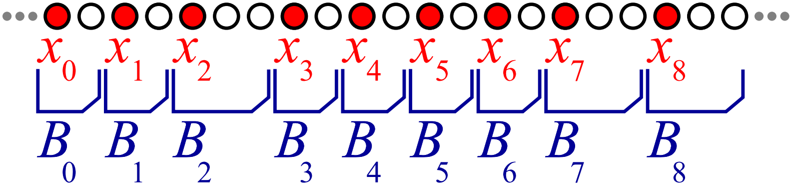
Elements are labeled $x_0, x_1, \dots, x_i, \dots$ (i modulo r).



Two Generators (Proof)

$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

Blocks are sets $B_k = \{x_k, x_k + 1, \ldots, x_{k+1} - 1\}$ (k modulo r).
("Intervals" closed on element x_k and open on x_{k+1})



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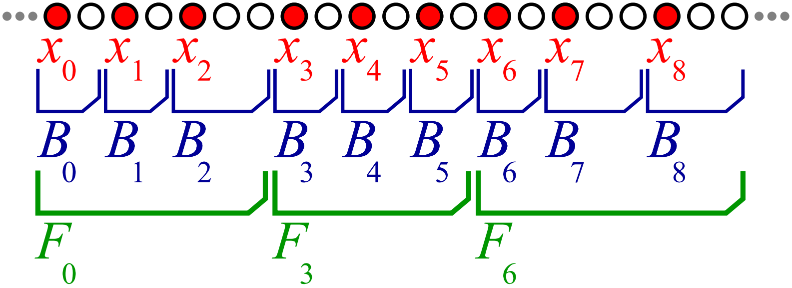
$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

Frames are collections $F_j = \{B_j, B_{j+1}, \ldots, B_{j+t-1}\}$ (j modulo r).
(There are t blocks in each frame.)



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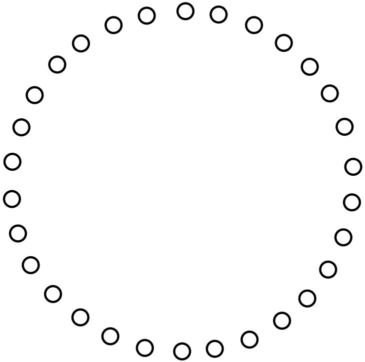

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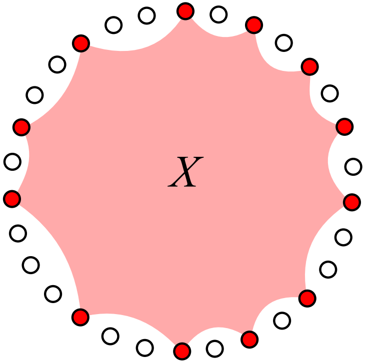

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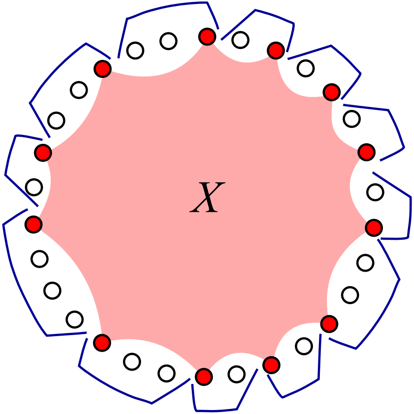

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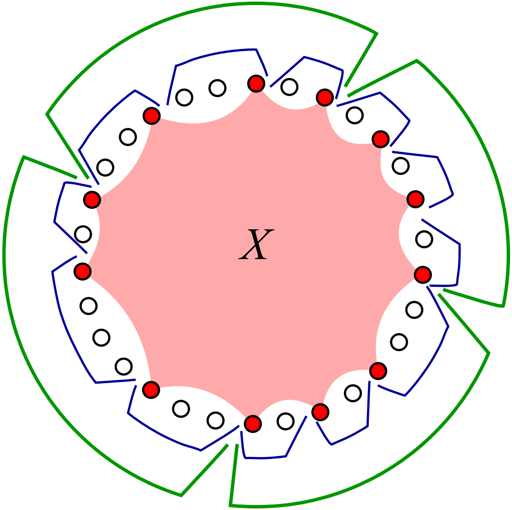

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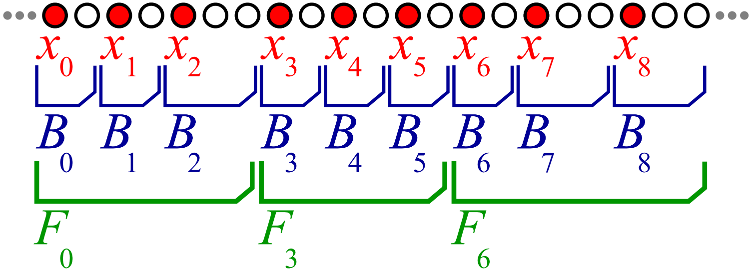
(a) Every block B_k has $|B_k| \geq 2$. (1 is a generator)

(b) Every frame F_j has a block $B_k \in F_j$ with $|B_k| \geq 3$.

(2t is a generator, so $x_{j+t} \neq x_j + 2t$.)



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Contradiction! $\therefore \omega(G) < r$.

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$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

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For unique saturation, we only need to check $G + \{0, 1\}$.

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Consider **frame family** $\mathcal{F}$

$$\mathcal{F} = \{F_1, F_{t+1}, F_{2t+1}, \dots, F_{r-t}\}, \qquad |\mathcal{F}| = 2t - 1.$$

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$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

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All blocks of of X (except B_0) have size 2 or 3.

There are exactly $(2t + 1)$ 3-blocks.

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$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

(a) There are at most $(t - 1)$ 2-blocks between 3-blocks.

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(c) If $B_{k_0}, B_{k_1}, \dots, B_{k_{2t}}$ be the 3-blocks.

$$k_0 \geq t - 1, \qquad k_{j+1} \in \{k_j + t - 2, k_j + t - 1\}, \qquad k_{2t} \leq r - t.$$

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$$k_0 \geq t-1, \qquad k_{j+1} \in \{k_j + t - 2, k_j + t - 1\}, \qquad k_{2t} \leq r - t.$$

A unique solution for $k_0, \dots, k_{2t}$: $k_{j+1} = k_j + t - 2$.

Defines X which is an r-clique. $\square$

Three Generators

Theorem (Hartke, S—) Let $t \geq 1$, $n = 9t^2 - 3t + 1$, and $r = 3t^2 - 2t + 1$. The Cayley complement $\overline{C}(\mathbb{Z}_n, \{1, 3t-1, 3t\})$ is r-primitive.

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t	S	r	n
1	$\{1,2,3\}$	2	7
2	$\{1,5,6\}$	9	31
3	$\{1,8,9\}$	22	73
4	$\{1,11,12\}$	41	133
5	$\{1,14,15\}$	66	211
6	$\{1,17,18\}$	97	307
7	$\{1,20,21\}$	134	421
8	$\{1,23,24\}$	177	553
9	$\{1,26,27\}$	226	703
10	$\{1,29,30\}$	281	871

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Pattern does <u>not</u> extend to $|S| \geq 4$!

More Generators

g	Generators	n	r
4	$\{1, 5, 8, 34\}$ $\{1, 11, 18, 34\}$	89	28
5	$\{1, 5, 14, 17, 25\}$	71	19
5	$\{1, 6, 14, 17, 36\}$	101	27
6	$\{1, 6, 16, 22, 35, 36\}$	97	21
7	$\{1, 20, 23, 26, 30, 32, 34\}$	71	15

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3. Is $\overline{C}(\Gamma, S)$ r-primitive for any group $\Gamma \not\cong \mathbb{Z}_n$?

Searching for uniquely saturated and strongly regular graphs with coupled augmentations[1]

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