Space-efficient algorithms for reachability in surface-embedded graphs

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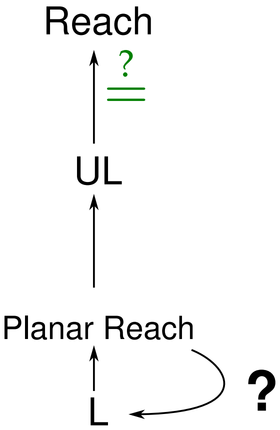
CCC 2012

June 29, 2012

Supported by NSF grants DMS-0354008, DMS-0914815, and CCF-0916525, and a CCC Student Travel Grant.


NL
L

$$\begin{tikzcd} \text{Reach} \\ \text{UL} \arrow[u, "\overset{?}{=}"'] \\ \text{Planar Reach} \arrow[u] \\ \text{L} \arrow[u] \end{tikzcd}$$


Reach
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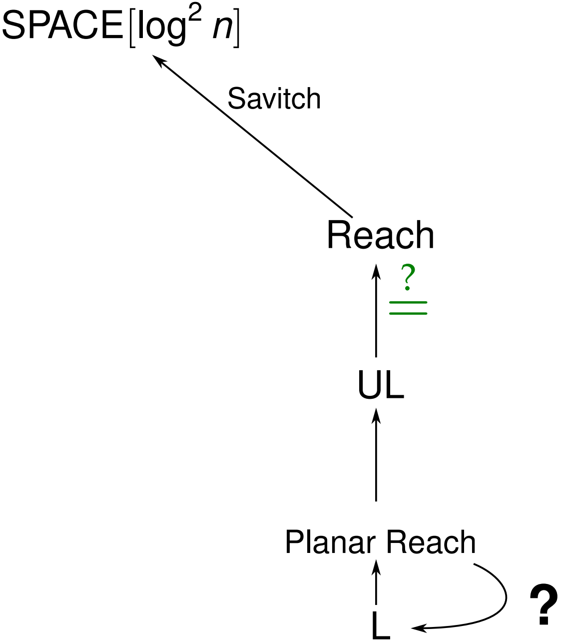

SPACE[log2 n]
Savitch
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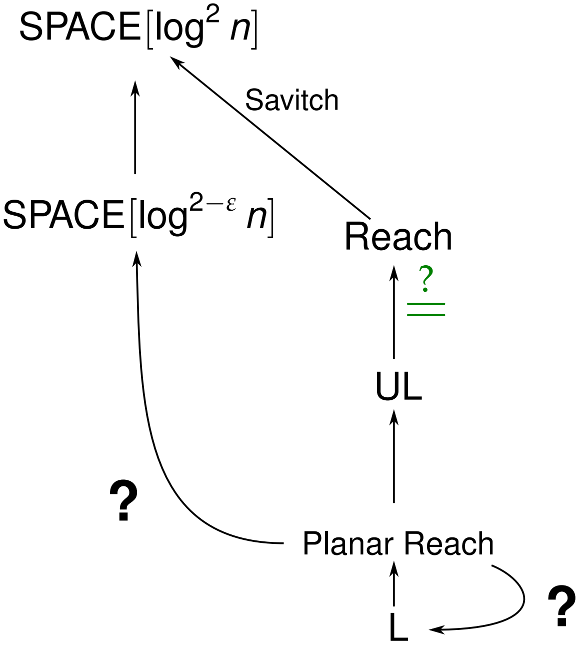

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Savitch
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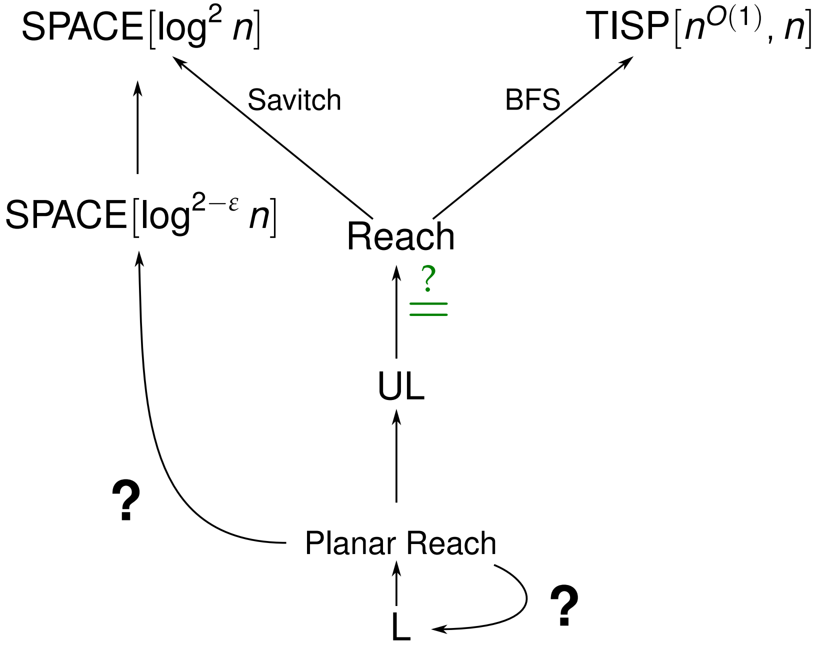

SPACE[log^2 n]
TISP[n^{O(1)}, n]
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BFS
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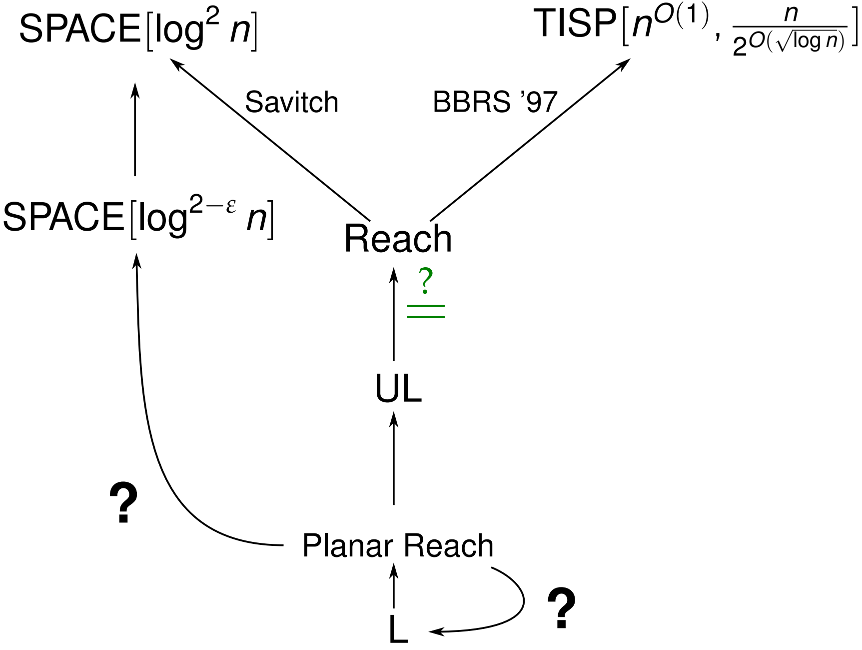

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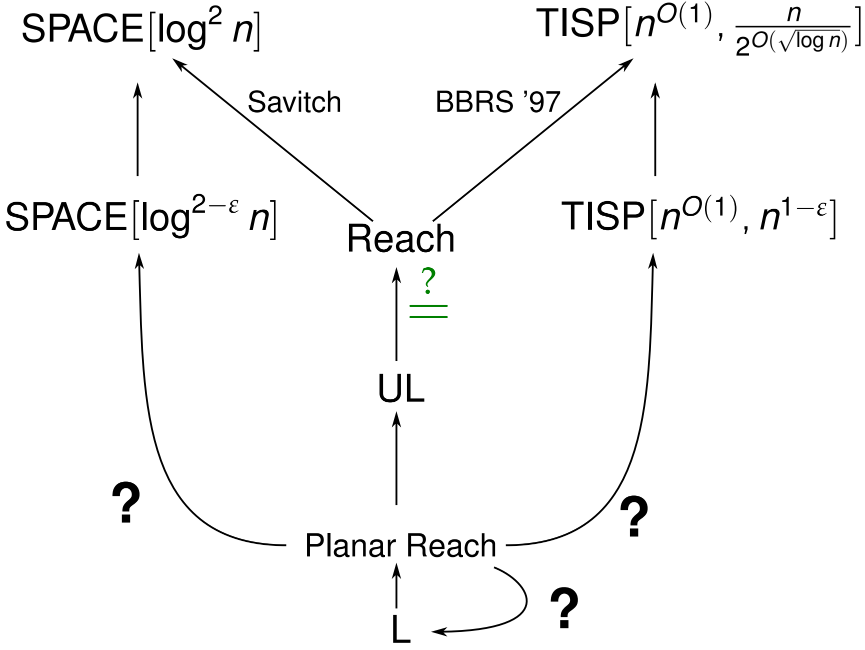

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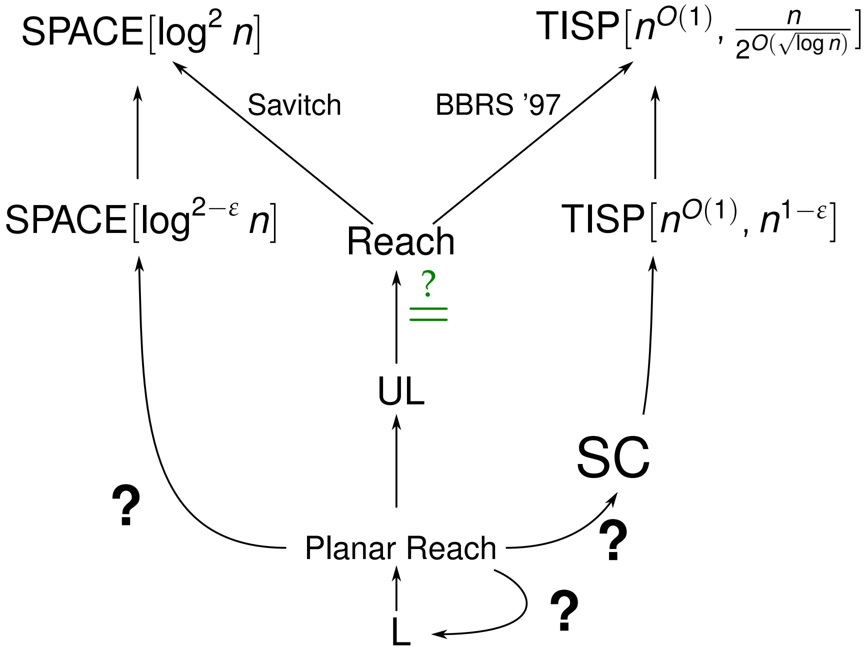

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Planar and Acyclic Restrictions

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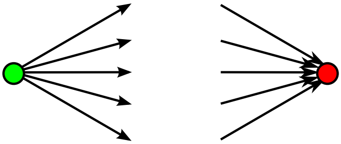

sources **sinks**

UNDIRECTED REACH in L

(Riengold, STOC 2005)

Planar + Acyclic Reachability in Log-Space

1. Series-parallel graphs
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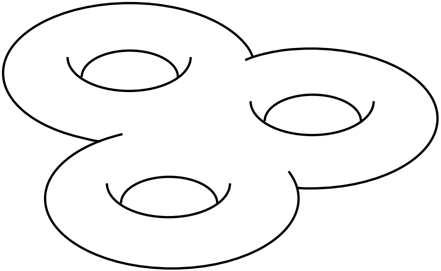
   1. $O(\log n + m)$-space algorithm for m sources.
   2. $O(\log n \cdot \log m)$-space algorithm for m sources.

Surface-embedded graphs

We also extend to graphs embedded in *surfaces of low genus*.

(orientable or non-orientable)



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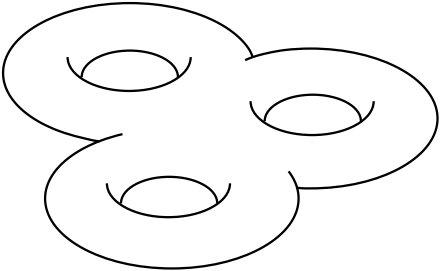

$\mathcal{G}(m, g)$ is the class of **acyclic** digraphs with at most m **sources** embedded in a surface of **genus at most** g.

Our Results (Stolee, Vinodchandran, '12)

Theorem (Sub-Savitch) Reachability for graphs of order n in $\mathcal{G}(m, g)$ is in

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$$\mathsf{TISP}[n^{O(1)}, \log n + m + g].$$

Reduction with Compression

Theorem (Stolee, Vinodchandran, '12) Given a graph $G \in \mathcal{G}(m, g)$ and $u, v \in V(G)$, we can compute in **log-space** a graph G' with vertices u', v' so that

1. There is a path from u to v in G **if and only if** there is a path from u' to v' in G'.
2. G' has $O(m + g)$ vertices.

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(Run Breadth-First-Search)

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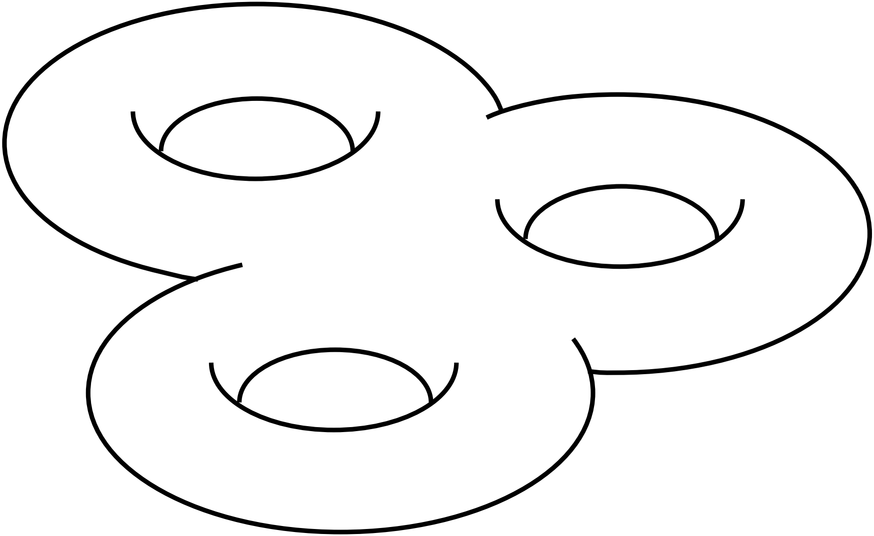
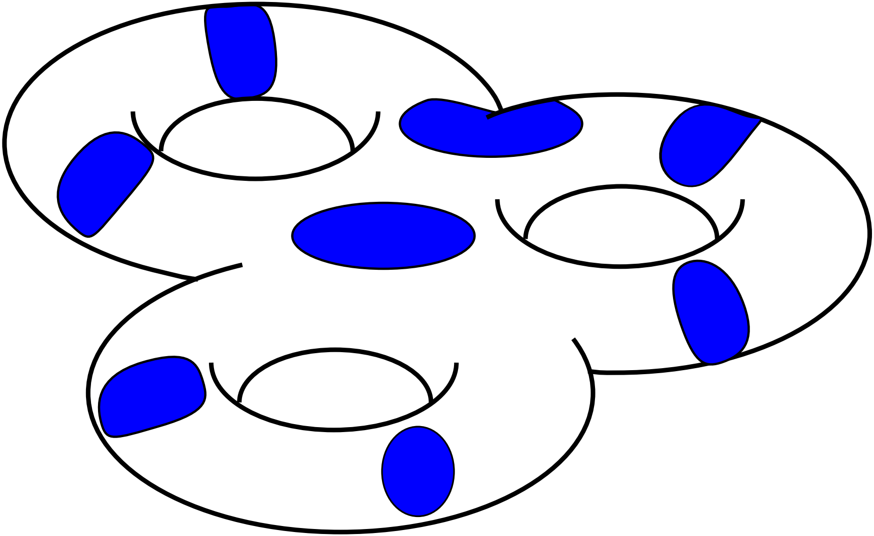
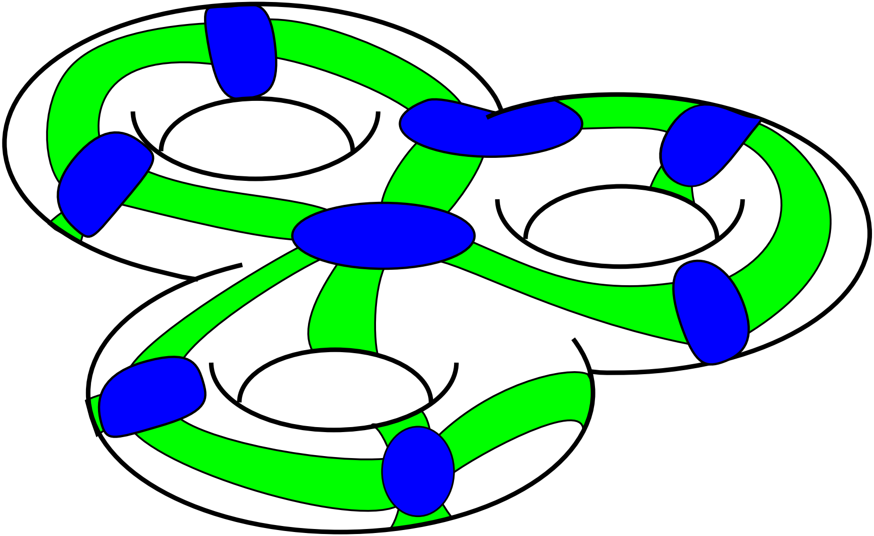
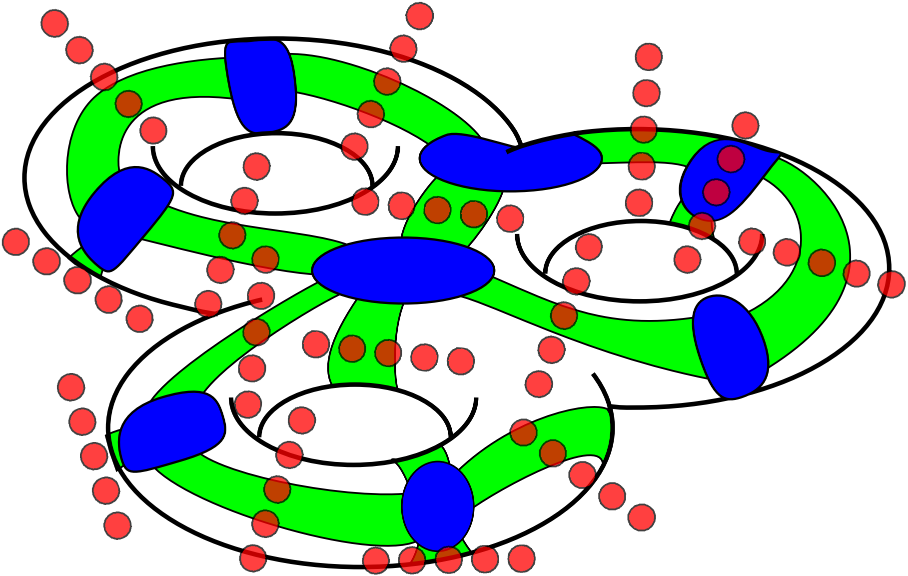
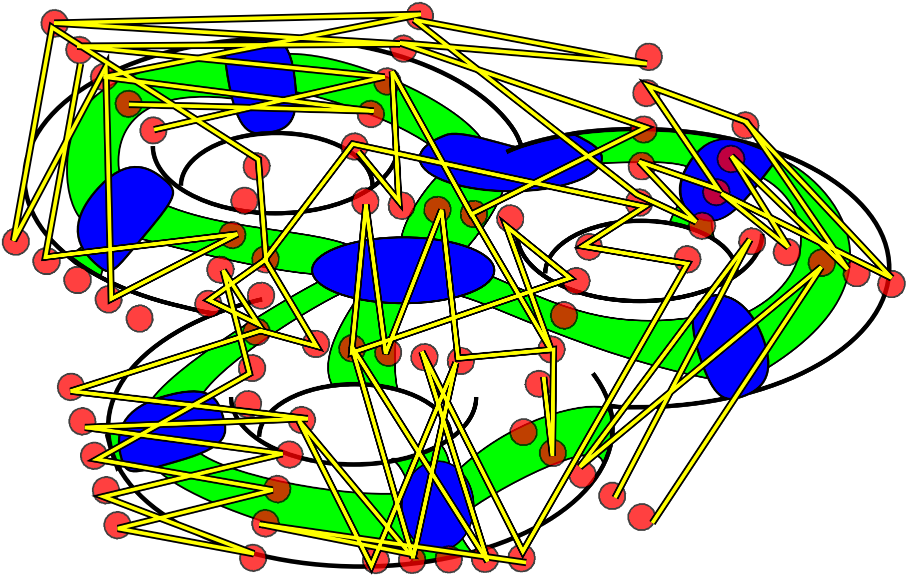
(Run BBRS)

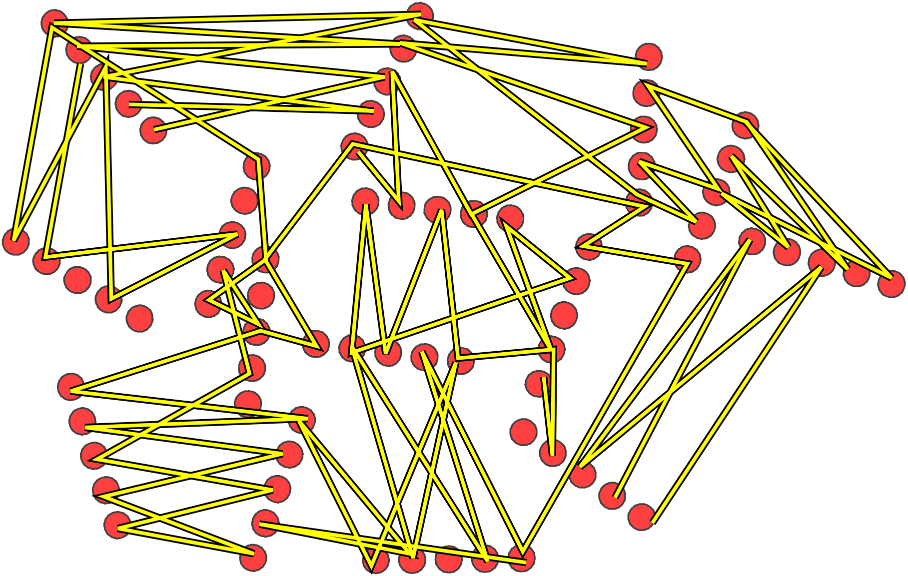












A Note About Embeddings

We take the embedding as **input**.

1. If $g = 0$ (G is planar), we can find an embedding in log-space.
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4. We **can** lower number of **sources** by increasing **genus**.

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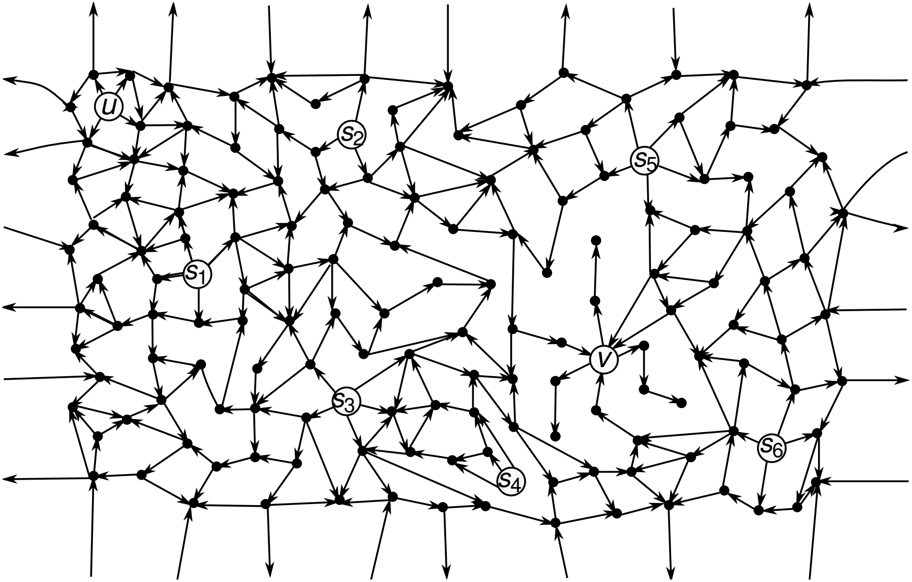
3. The connected components are trees rooted at $s_1, \ldots, s_m, u, v$:

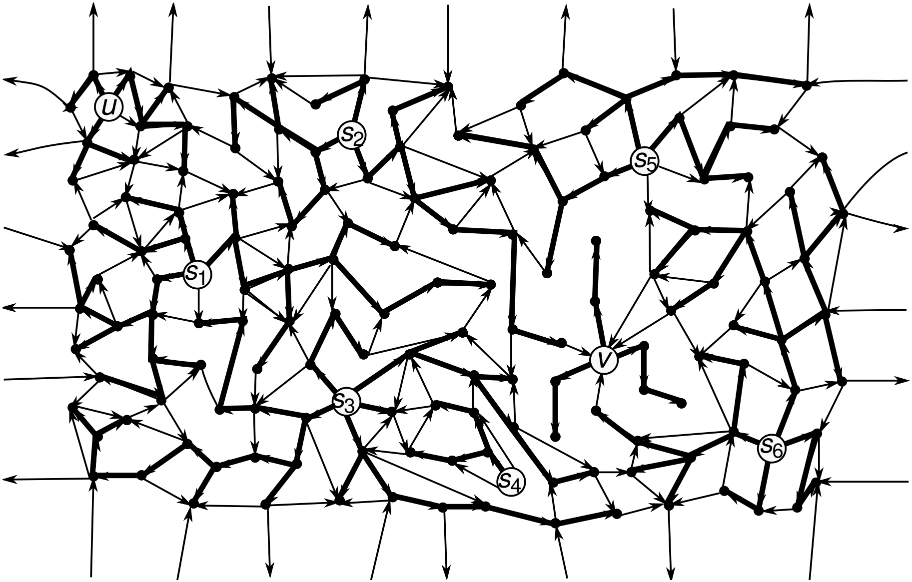
$$T_{s_1}, \ldots, T_{s_m}, T_u, T_v.$$

(We can remove vertices in T_v)

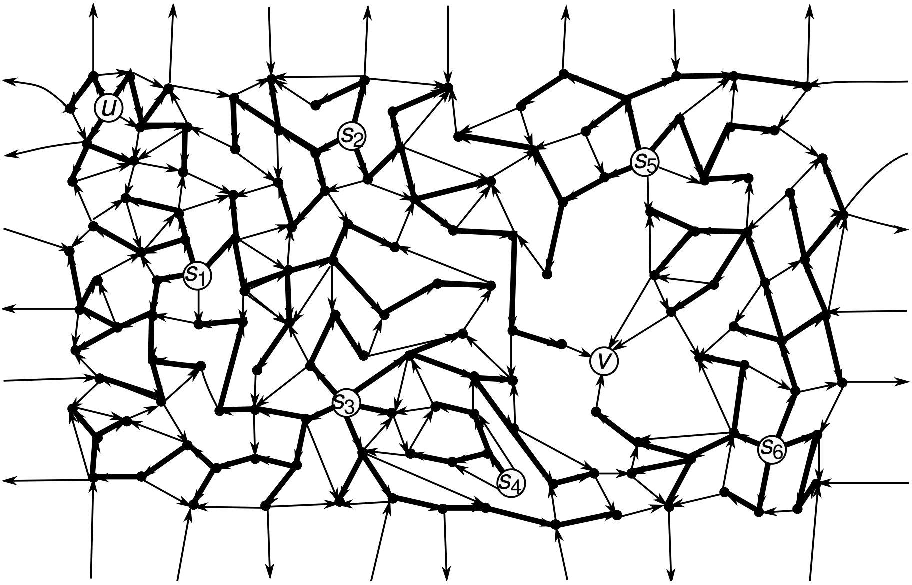
Start with G (Here on the torus).



Select Tree Edges.



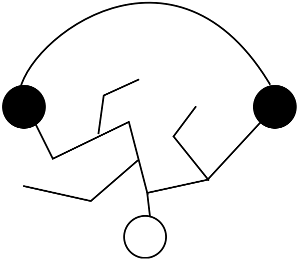
Remove vertices in v's tree.



Local Edges

An edge $x \to y$ is **local** if

1. x and y are within the same source tree T_{s_i}, and
2. The (undirected) tree path from x to y along with the edge xy is a **contractible cycle**.



Contractible Cycles

A cycle is **contractible** if:

1. It partitions $s_1, \ldots, s_m, u, v$ *trivially*.
2. The trivial part of the surface is homeomorphic to a disk.

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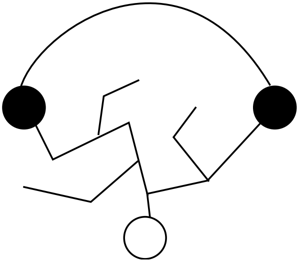

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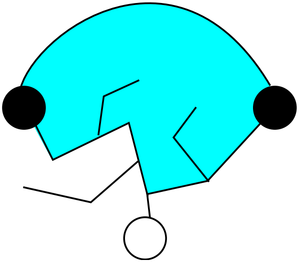
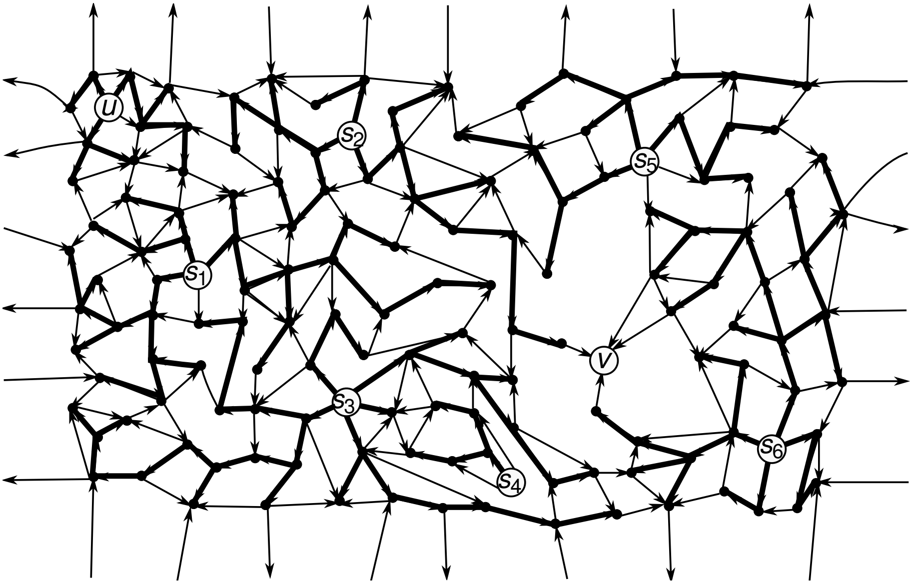


Identify Tree Edges.



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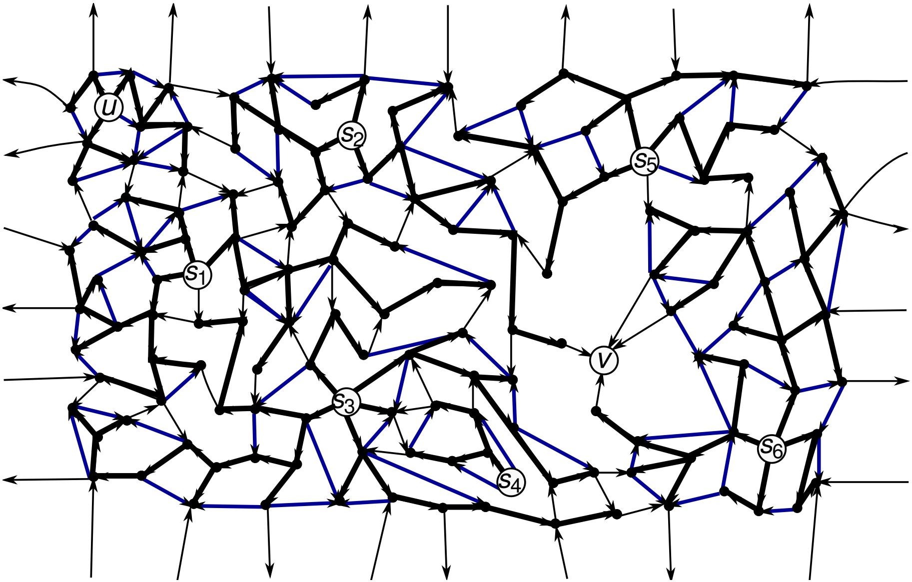
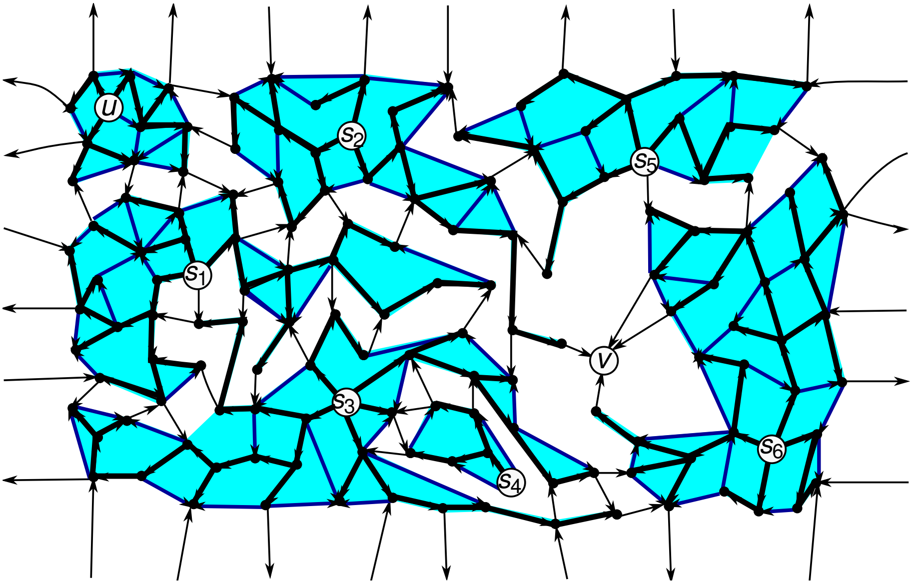
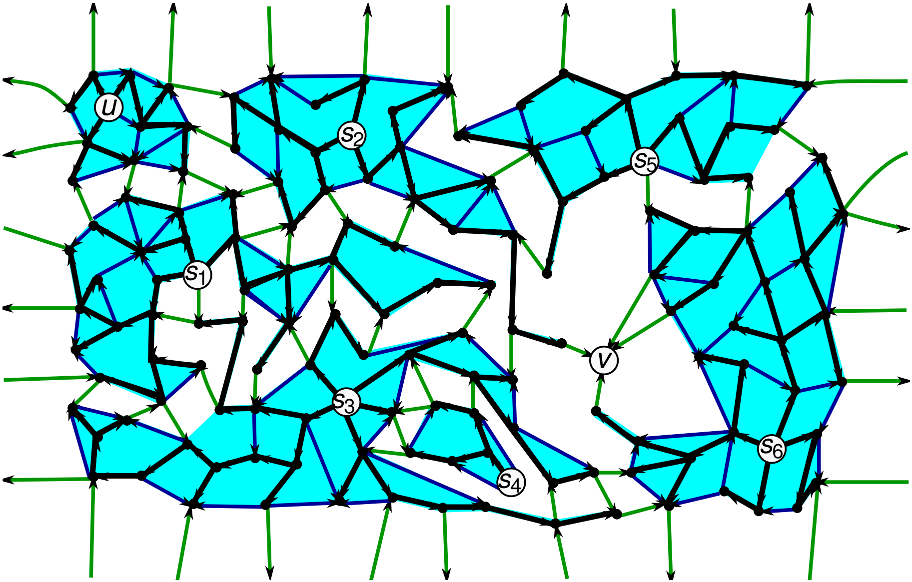


Mark tree regions $\mathcal{R}[T]$.



Identify **global edges**.



Topological Equivalence

Two global edges *xy* and *wz* are **topologically equivalent** if

1. The trees *xy* and *wz* span are the same, and
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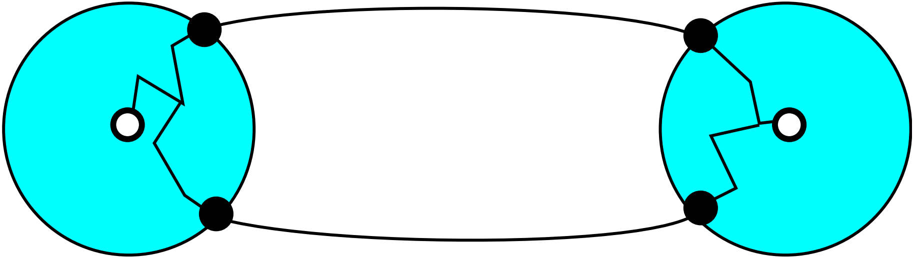

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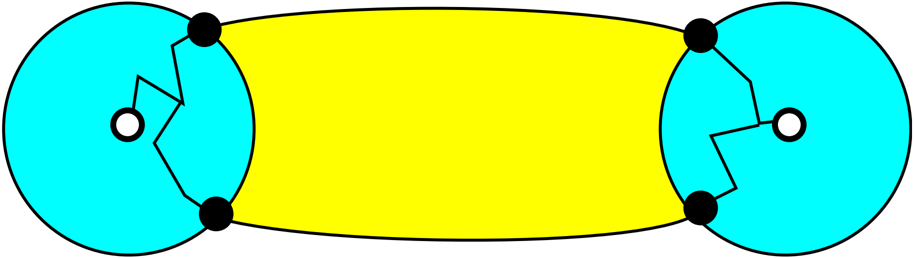

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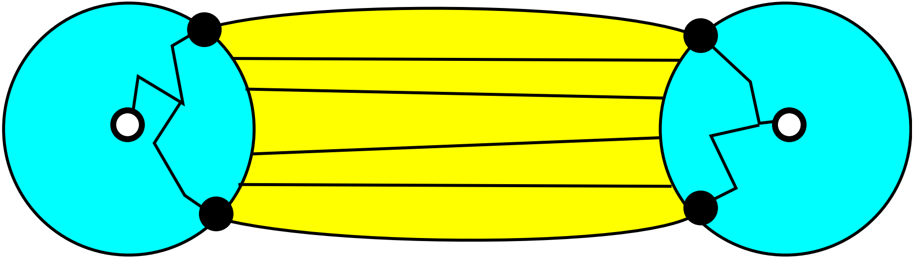

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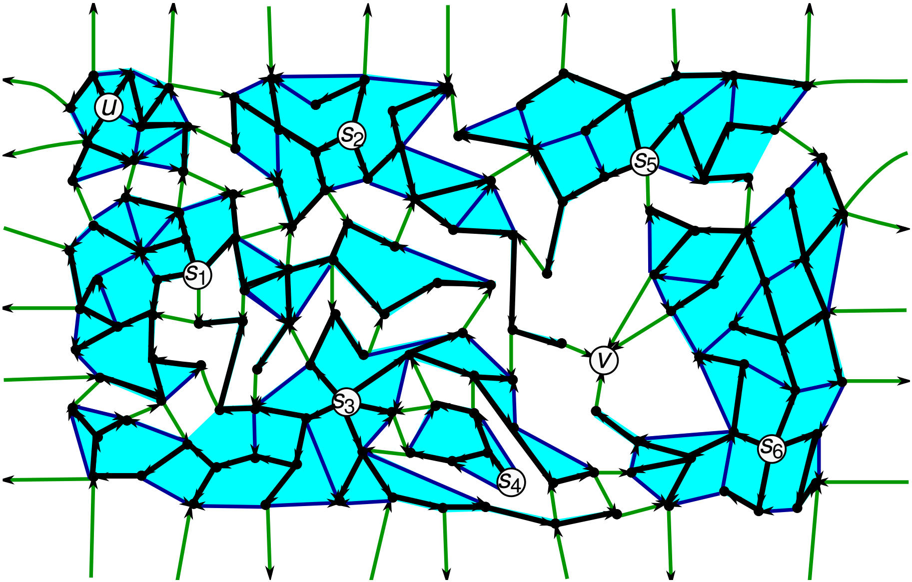
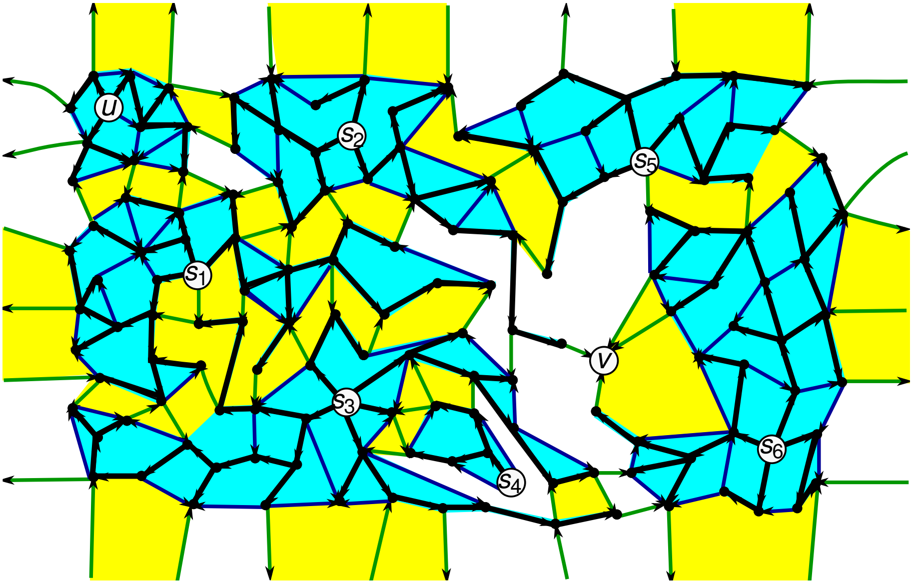
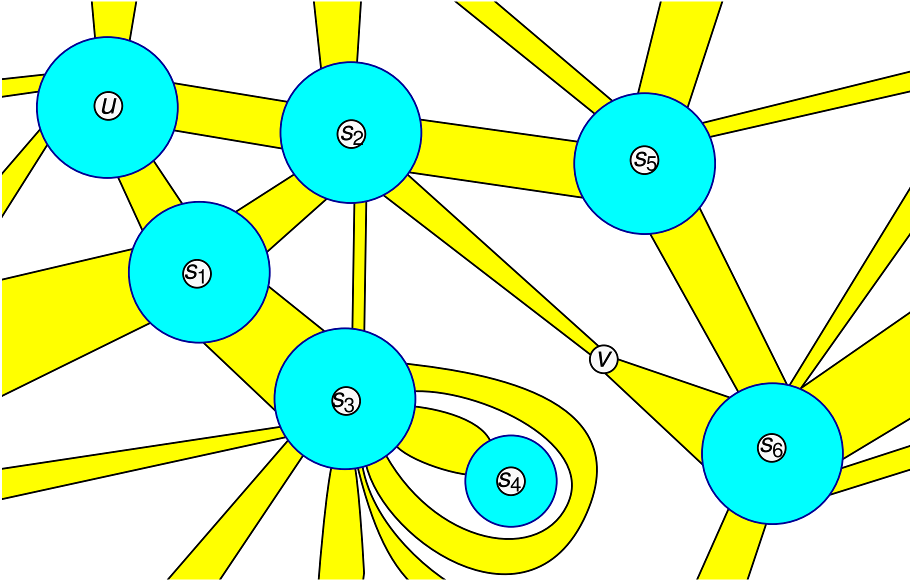


Mark equivalence class regions $\mathcal{R}[E]$.



A compressed view.



Number of Equivalence Classes

A simple application of **Euler's Formula**:

$$n - e + f = \begin{cases} 2 - 2g & \text{(orientable)} \\ 2 - g & \text{(non-orientable)} \end{cases}$$

shows that the number of equivalence classes is $O(m + g)$.

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We will "blow up" these equivalence classes to form our vertices of G'.

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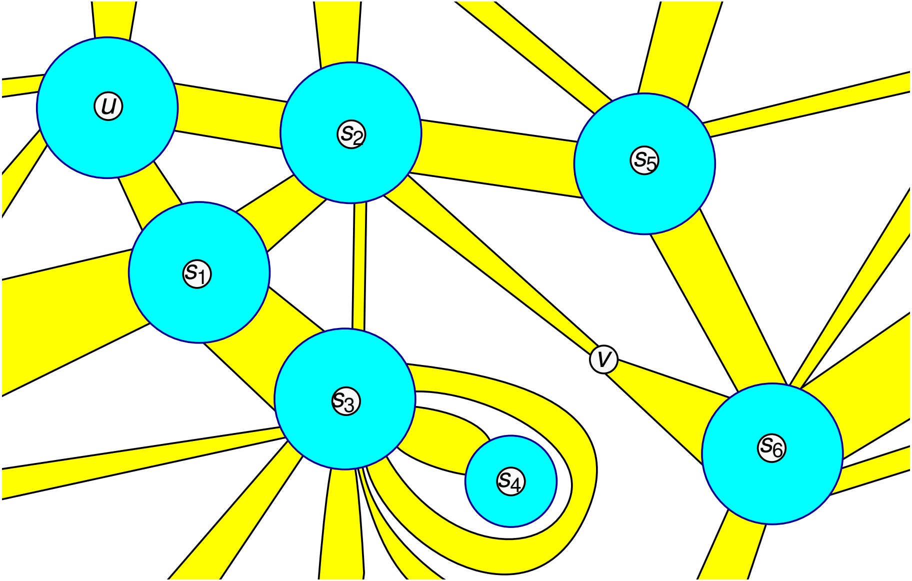
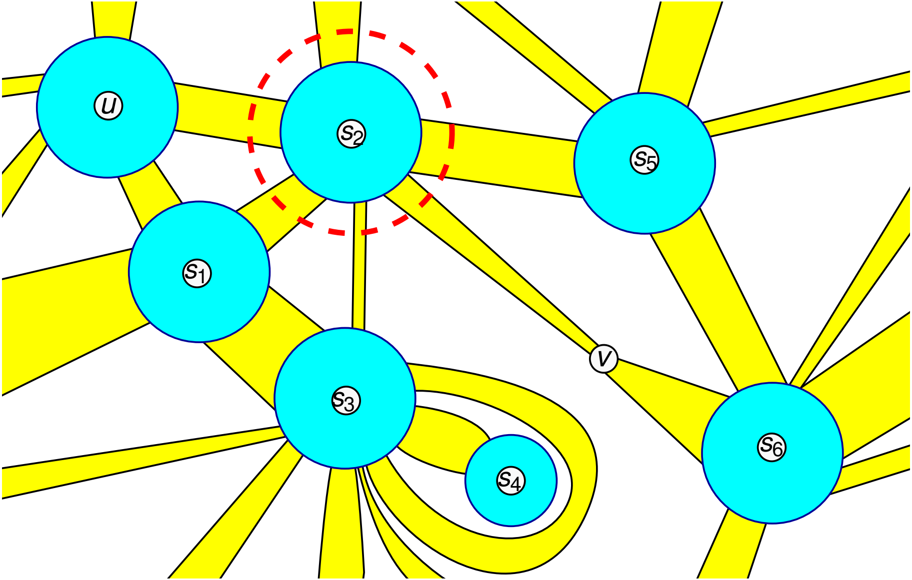
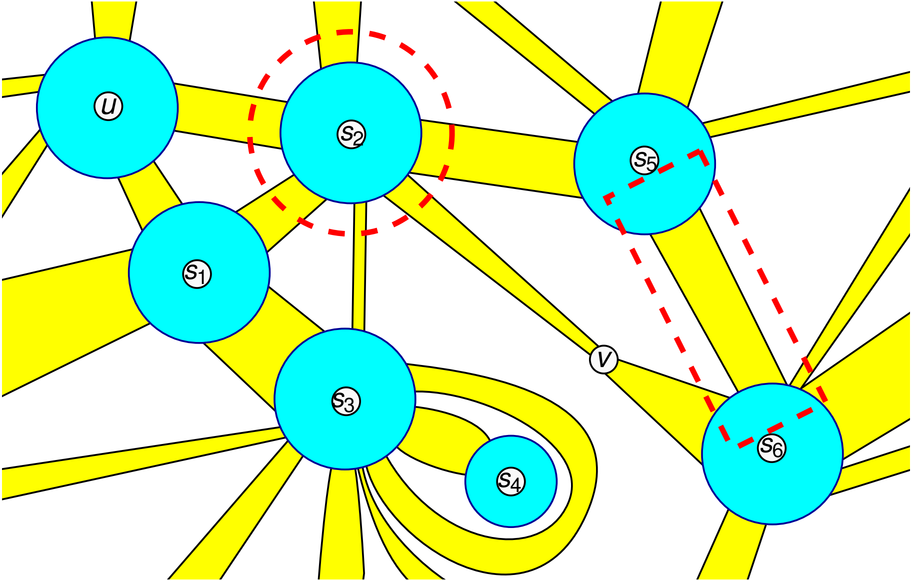


1. Local reachability within a tree.



2. Global reachability within an equivalence class.



Irreducible Paths

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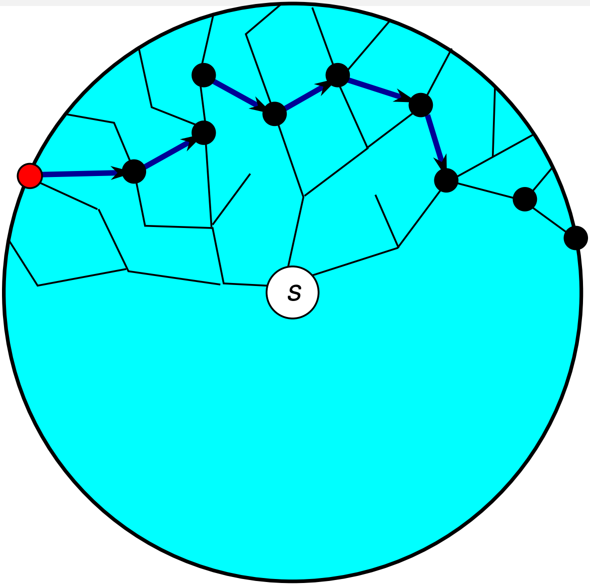
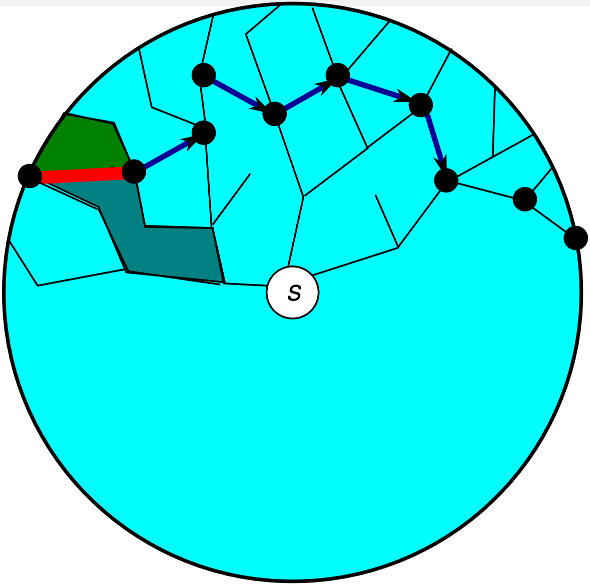
Irreducible paths are **nice** because they follow a single **clockwise** or **counterclockwise** direction while traveling through a source tree.

Irreducible Paths in a Source Tree



Irreducible Paths in a Source Tree



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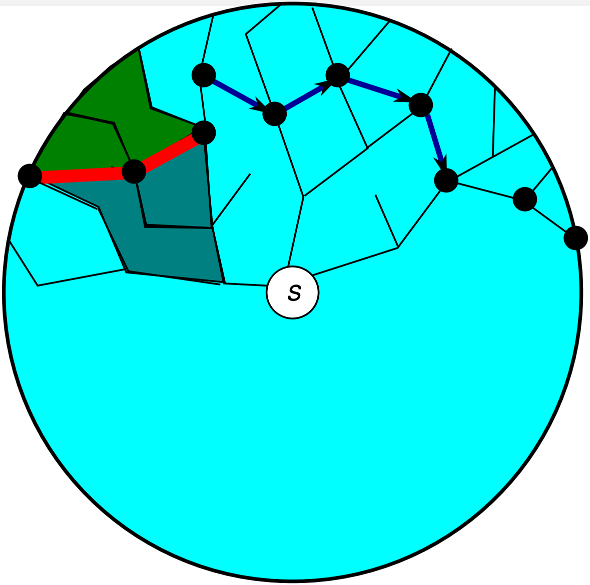

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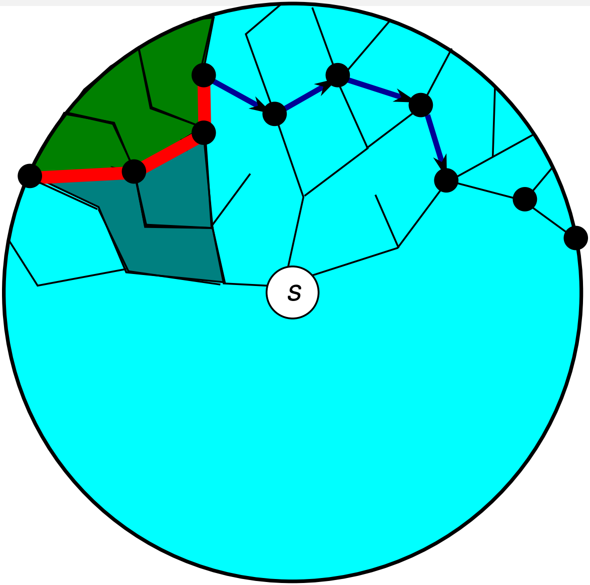

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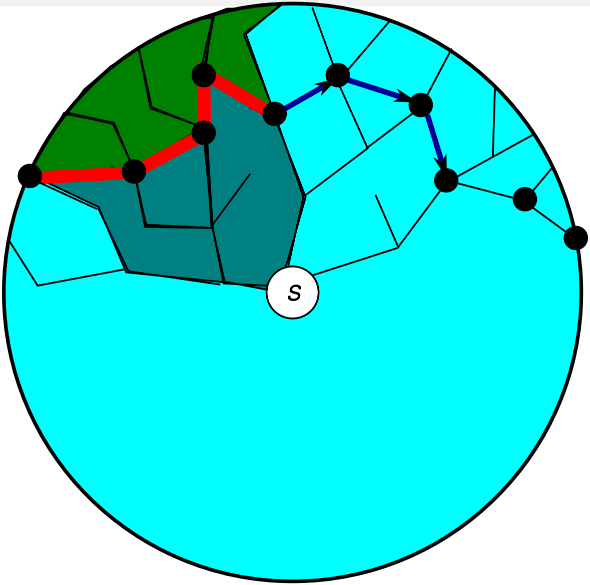

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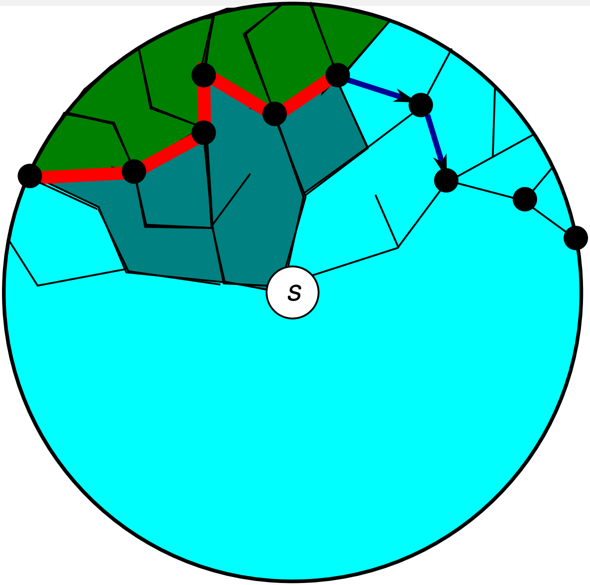

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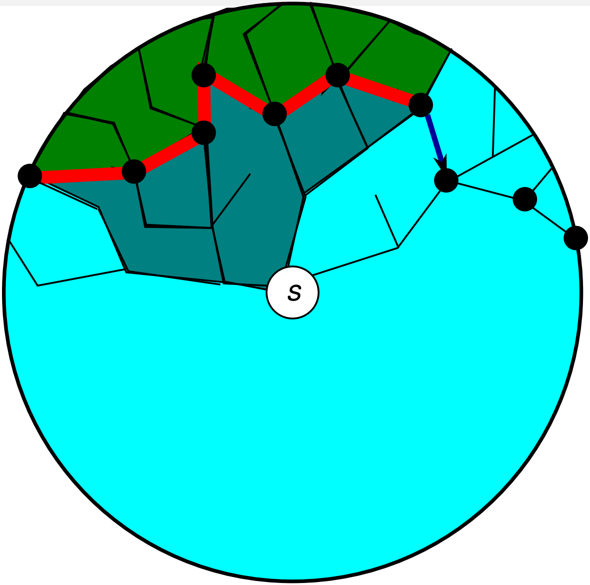

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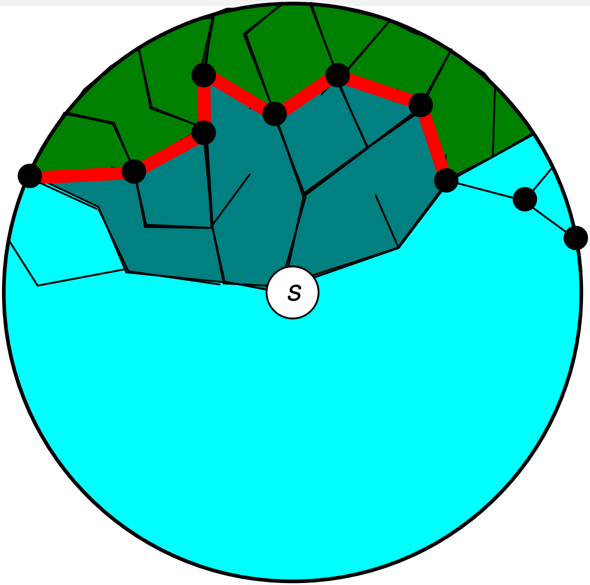

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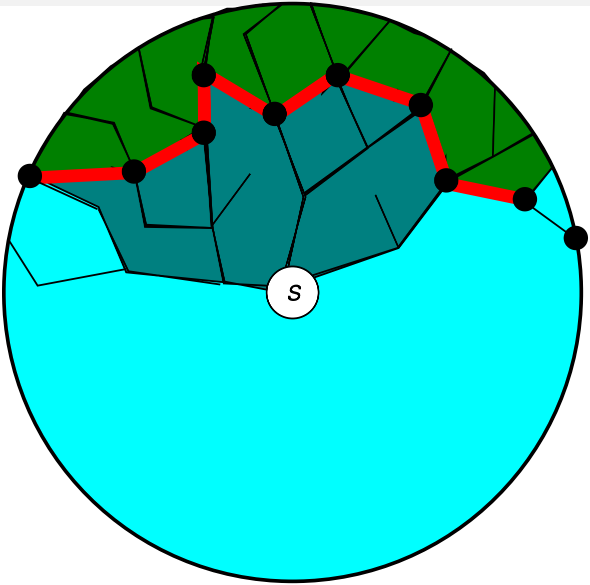

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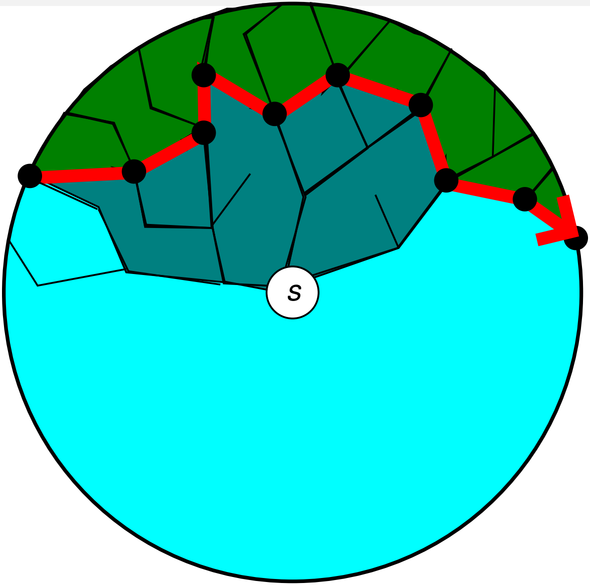

Directional Reachability Within Source Trees

The tree and local edges within a source tree, embedded on the region $\mathcal{R}[T]$ is a *single-source, multiple-sink, planar DAG*.

We can use ABCDR's algorithm as a black box (almost) to find **directional reachability** within source trees.

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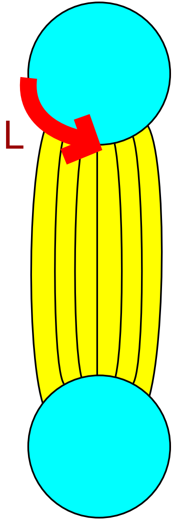
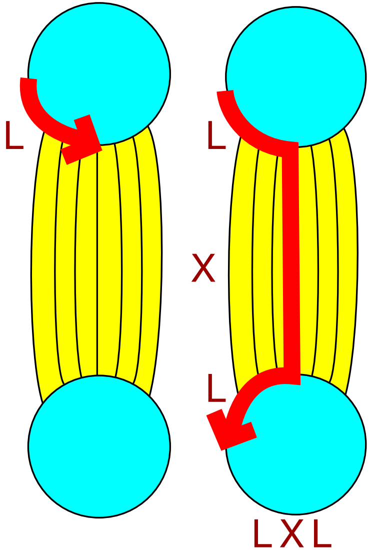
Now, what happens in global edges?

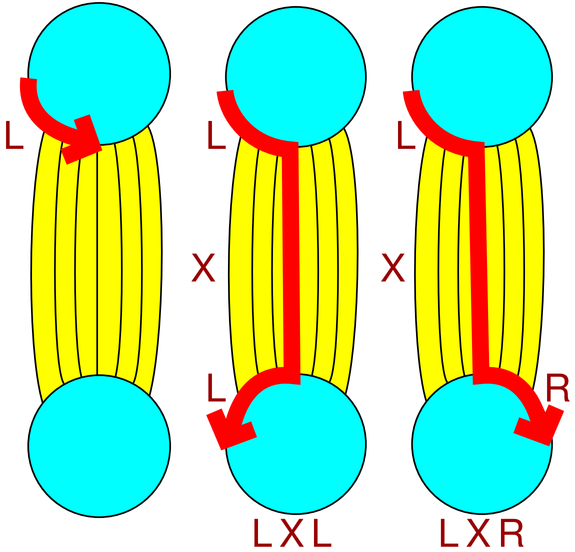
Patterns on an Equivalence Class



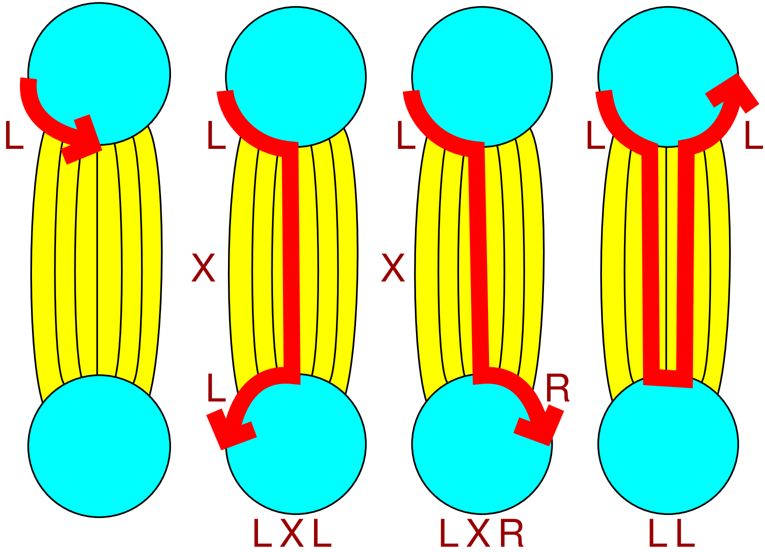
Patterns on an Equivalence Class



Patterns on an Equivalence Class



Patterns on an Equivalence Class



Pattern Descriptions

$$\mathcal{P} = \{\langle \mathsf{L\,X\,L}\rangle, \langle \mathsf{R\,X\,R}\rangle, \langle \mathsf{L\,X\,R}\rangle, \langle \mathsf{R\,X\,L}\rangle, \langle \mathsf{L\,L}\rangle, \langle \mathsf{R\,R}\rangle\}$$

A **pattern node** (denoted **x**, **y**, or **z**) consists of:

1. An equivalence class index i (for the ith class)
2. A pattern from $\mathcal{P}$
3. An entrance tree (A or B)
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Subtle: Requires $O(\log(m+g))$ bits to describe a pattern node.

Pattern Descriptions

$$\mathcal{P} = \{\underbrace{\langle \mathrm{L\,X\,L}\rangle, \langle \mathrm{R\,X\,R}\rangle}_{\text{Nesting}}, \underbrace{\langle \mathrm{L\,X\,R}\rangle, \langle \mathrm{R\,X\,L}\rangle, \langle \mathrm{L\,L}\rangle, \langle \mathrm{R\,R}\rangle}_{\text{Full}}\}$$

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Subtle: Requires $O(\log(m+g))$ bits to describe a pattern node.

Structure of Full Patterns

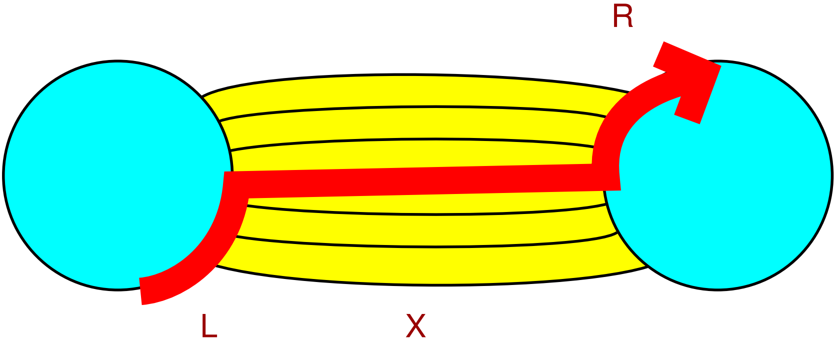
For a pattern node $\mathbf{x}$ that uses a *full pattern* over an equivalence class E_i, there are two edges:

$$e_{\mathbf{x}}^{\text{in}} \qquad \text{and} \qquad e_{\mathbf{x}}^{\text{out}}$$

so that a vertex w has an irreducible path using tree, local, and E_i edges inducing $\mathbf{x}$, then

1. w can reach $e_{\mathbf{x}}^{\text{in}}$, and
2. everything w can reach with such paths is reachable from $e_{\mathbf{x}}^{\text{out}}$.

Structure of Full Patterns



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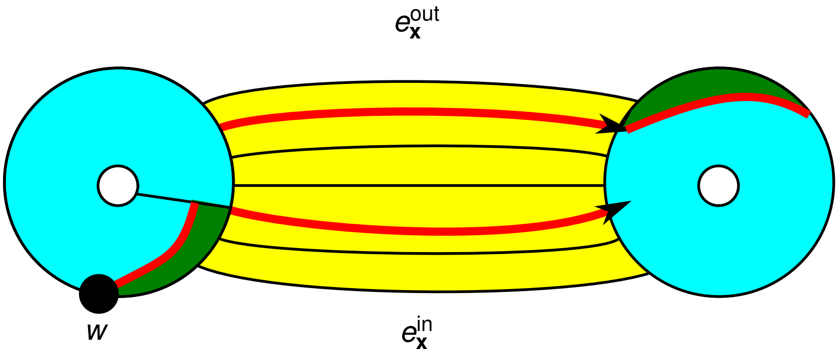


Intuition

For two pattern descriptions **x** and **y** with matching exit-entrance, we want:

Put an edge $\mathbf{x} \rightarrow \mathbf{y}$

$$\Longleftrightarrow$$

there is a local path from $e_{\mathbf{x}}^{\mathrm{out}}$ to $e_{\mathbf{y}}^{\mathrm{in}}$.

Intuition

For two pattern descriptions **x** and **y** with matching exit-entrance, we want:

Put an edge $\mathbf{x} \to \mathbf{y}$

$$\Longleftrightarrow$$

there is a local path from $e^{\text{out}}_{\mathbf{x}}$ to $e^{\text{in}}_{\mathbf{y}}$.

But nesting patterns mess this up!

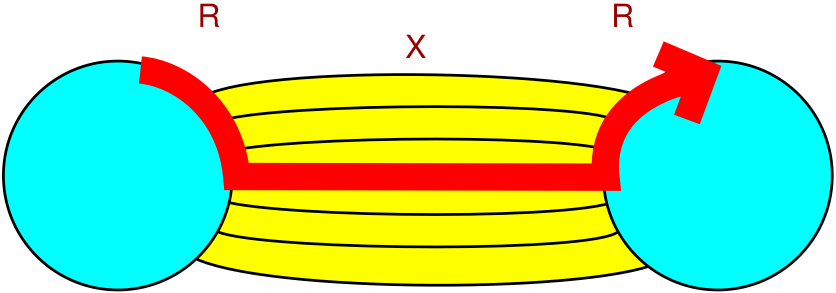
Structure of Nesting Patterns

For a pattern node $\mathbf{x}$ that uses a *nesting pattern* over an equivalence class E_i, we have

$$e_{\mathbf{x}}^{\text{in}} \qquad = \qquad e_{\mathbf{x}}^{\text{out}}$$

BUT: A vertex w can use the pattern without reaching $e_{\mathbf{x}}^{\text{in}}$!

Structure of Nesting Patterns



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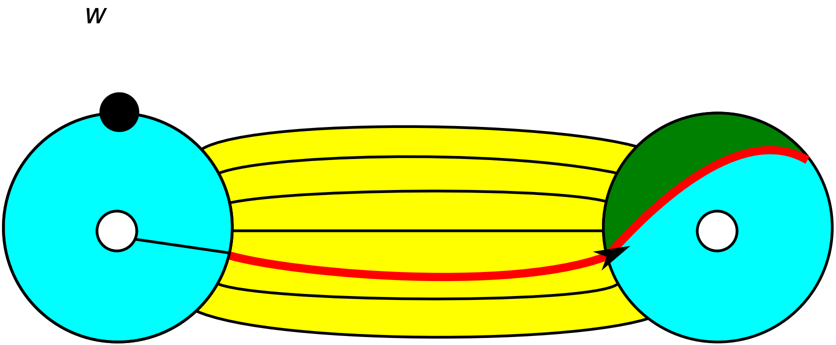

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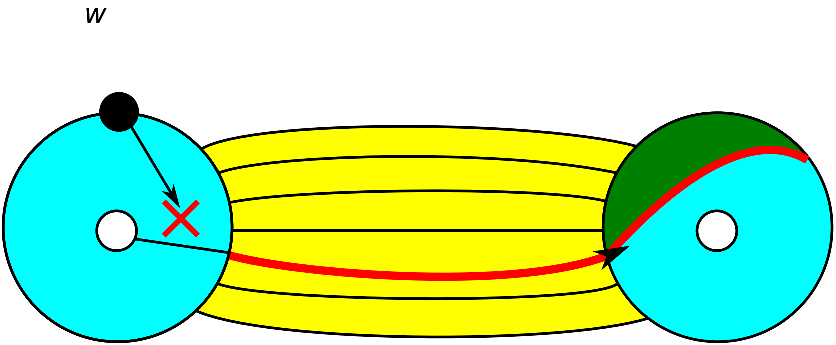

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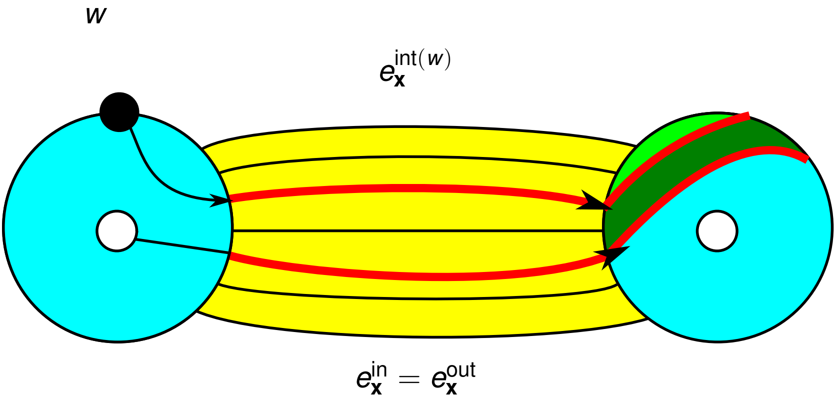
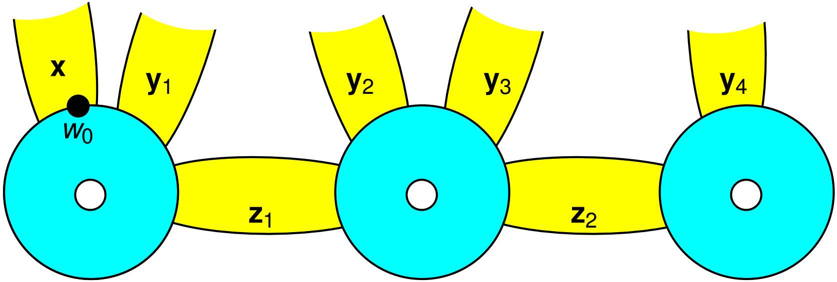


Adjacency Certificates

For two pattern descriptions $\mathbf{x}$, $\mathbf{y}$, place an edge $\mathbf{x} \to \mathbf{y}$ if and only if there is an **adjacency certificate** $\mathbf{z}_1, \ldots, \mathbf{z}_k$ of nesting patterns so that

1. Define $w_0 = \text{Head}(e_{\mathbf{x}}^{\text{out}})$ and $w_{j+1} = \text{Head}(e_{\mathbf{z}_{j+1}}^{\text{int}(w_j)})$.
2. For all $j \in \{0, \ldots, k-1\}$, the vertex w_j cannot reach $e_{\mathbf{z}_{j+1}}^{\text{in}}$ via local paths.
3. The vertex w_k can reach $e_{\mathbf{y}}^{\text{in}}$ via a local path.

Adjacency Certificates



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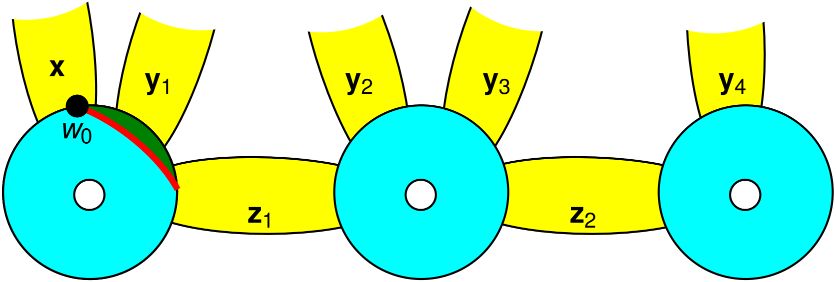

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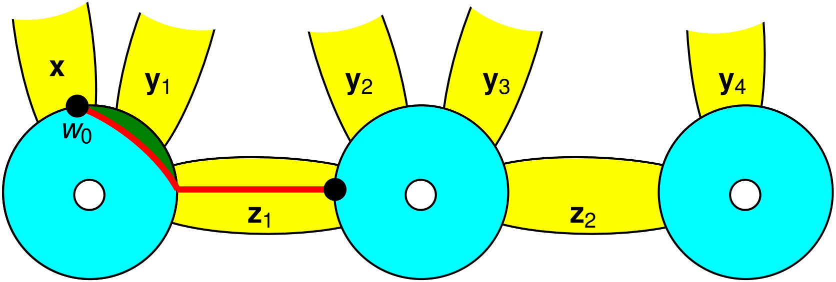

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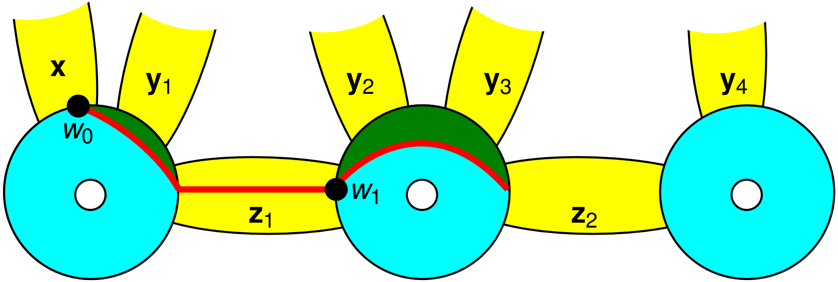

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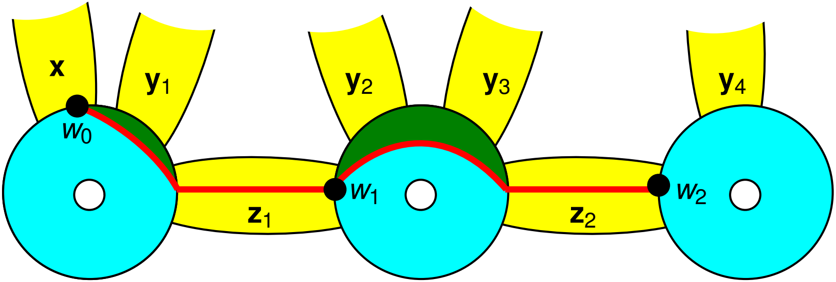

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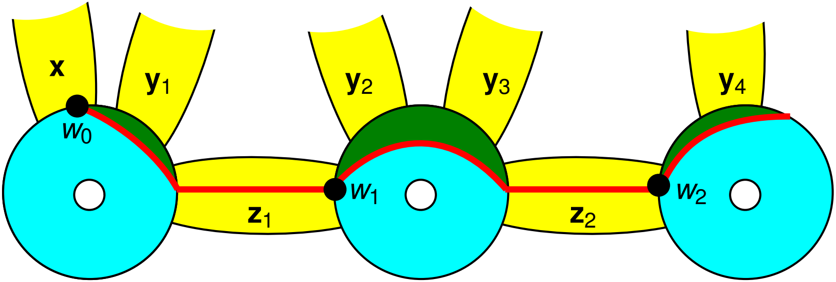

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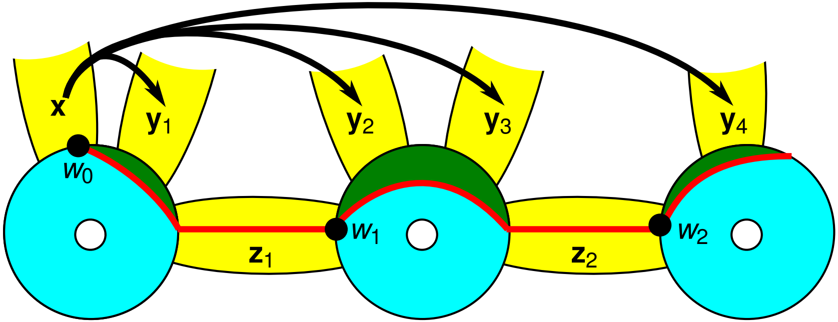

Special Vertices

Two special vertices: u' and v'.

1. $u' \to \mathbf{x}$ if and only if $\mathbf{x}$ is a pattern description over an equivalence class incident to T_u *and* starts on T_u.
2. $\mathbf{x} \to v'$ if and only if $\mathbf{x}$ is a pattern description over an equivalence class incident to T_v *and* ends on T_v.

Now, there is a path from u to v in G **if and only if** there is a path from u' to v' in G'!

Main Theorem

Theorem (Stolee, Vinodchandran, '12) Given a graph $G \in \mathcal{G}(m, g)$ and $u, v \in V(G)$, we can compute in **log-space** a graph G' with vertices u', v' so that

1. There is a path from u to v in G **if and only if** there is a path from u' to v' in G'.
2. G' has $O(m + g)$ vertices.

Future Directions

Extend this construction:

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Completely new directions?

Space-efficient algorithms for reachability in surface-embedded graphs

Derrick Stolee* N. V. Vinodchandran

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CCC 2012
June 29, 2012

Supported by NSF grants DMS-0354008, DMS-0914815, and CCF-0916525, and a CCC Student Travel Grant.