Computational Combinatorics Blog

`http://computationalcombinatorics.wordpress.com/`

Some topics:

- Using computational software as black box.
- Isomorph-free generation.
- Canonical labelings, orbit calculations.
- Orbital branching. (on the way)
- Flag Algebras. (on the way)
- Local search techniques (on the way)
- More...

Guest authors are requested!

Ordered Ramsey Theory and Track Representations of Graphs

Kevin G. Milans   Derrick Stolee*   Douglas B. West

University of Illinois
stolee@illinois.edu
http://www.math.uiuc.edu/∼stolee/

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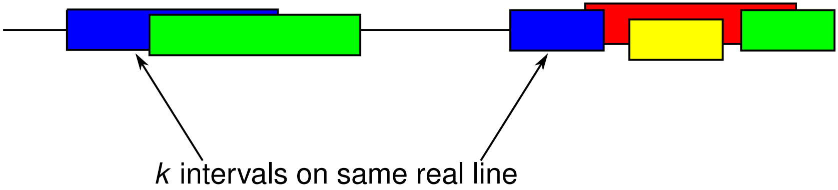

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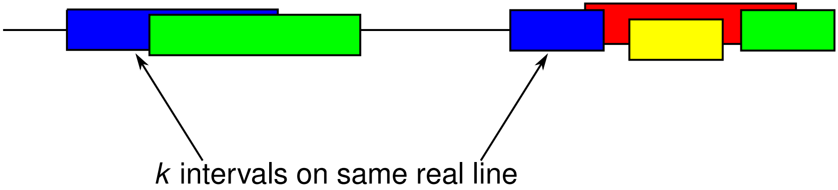

If $i(G) = 1$, G is an **interval graph**.

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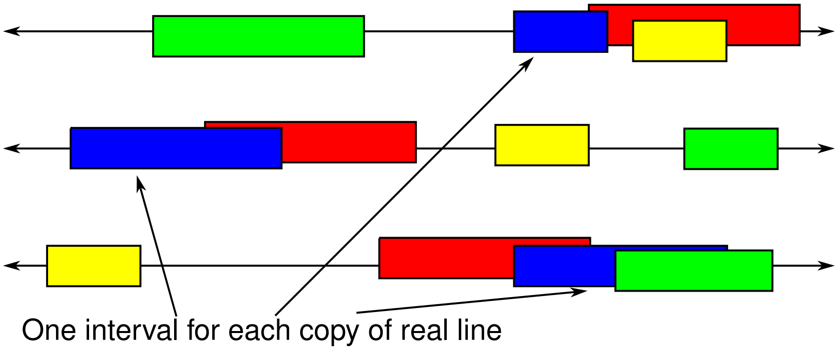
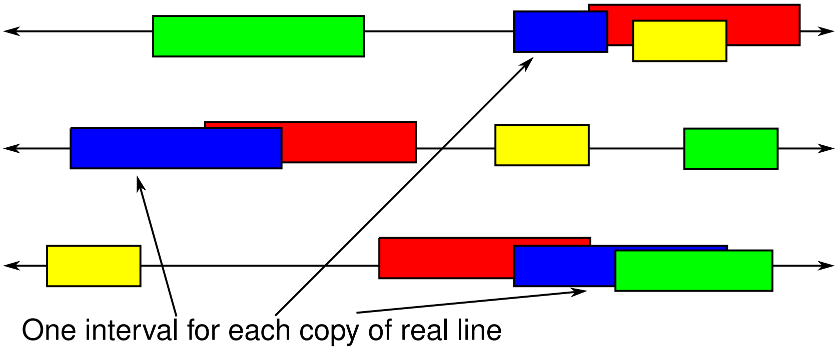


Track Number

Let $\tau(G)$ be the minimum number *t* such that *G* has a ***t*-track representation**.



If $\tau(G) = 1$ then $i(G) = 1$ and *G* is an interval graph.

Interval and Numbers

The track number is at least the interval number:

$$i(G) \leq \tau(G)$$

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The interval number of line graphs is 2:

$$i(L(G)) = 2.$$

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We prove (3).

Asymptotics of $\tau(L(K_n))$

Theorem (Milans, Stolee, West, 2012+)

$$\Omega\left(\frac{\lg\lg n}{\lg\lg\lg n}\right) \leq \tau(L(K_n)) \leq O\left(\lg\lg n\right).$$

The Lower Bound

Let $\mathrm{R}_t(K_6^3)$ denote the t-color Ramsey number for the 3-uniform hypergraph of order 6.

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By Conlon, Fox, and Sudakov, $\mathrm{R}_t(K_6^3) \leq 2^{2^{(4+o(1))t \lg t}}$ and therefore when $\lg \lg n \geq 5t \lg t$, we have $\tau(L(K_n)) > t$.

If $n \geq \mathrm{R}_t(K_6^3)$, then $\tau(L(K_n)) > t$

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For a triple $x < y < z$, the edges xy and yz are adjacent in $L(K_n)$, so some track i has the intervals for xy and yz intersecting.

This presents a t-coloring of the triples over $[n]$.

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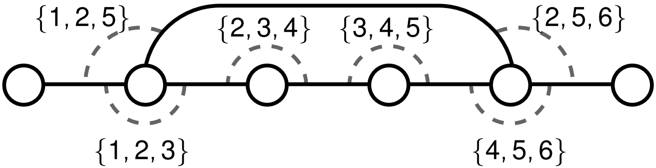
This presents a t-coloring of the triples over $[n]$.

Since $n > \mathrm{R}_t(K_6^3)$, there is a set of 6 elements $v_1 < \cdots < v_6$ that induce a monochromatic copy of K_6^3.

If $n \geq \mathrm{R}_t(K_6^3)$, then $\tau(L(K_n)) > t$



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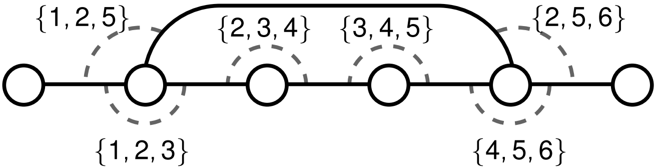

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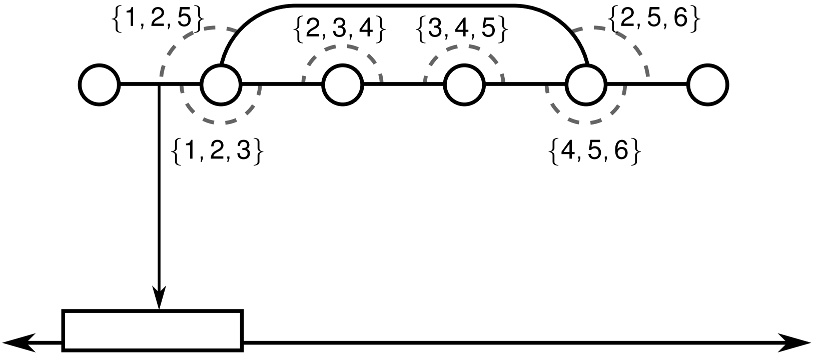

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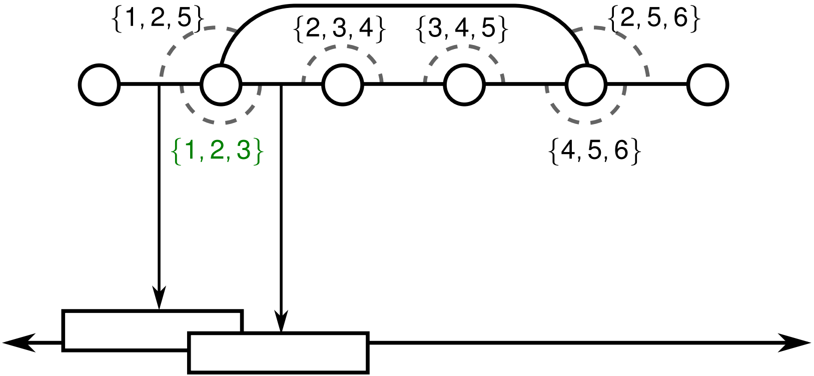

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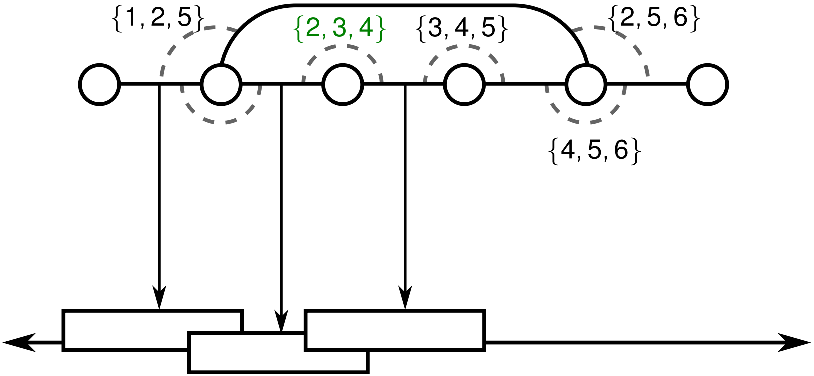

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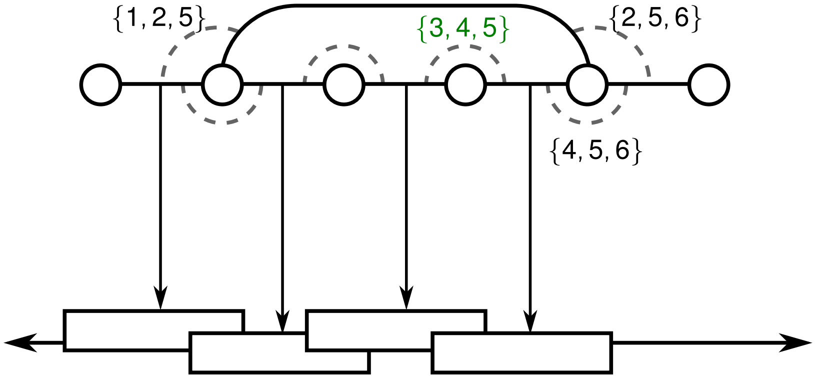

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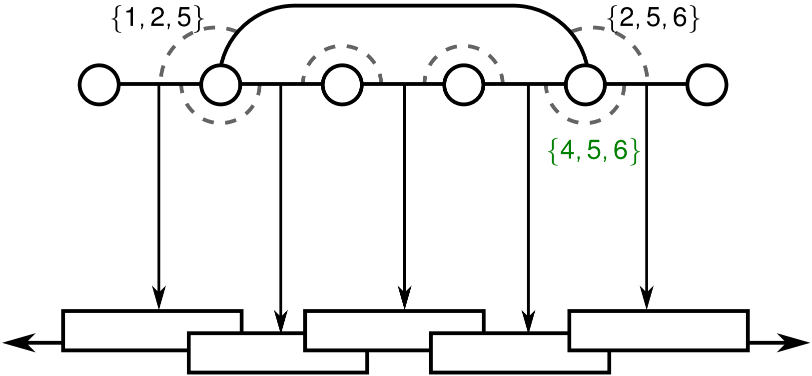

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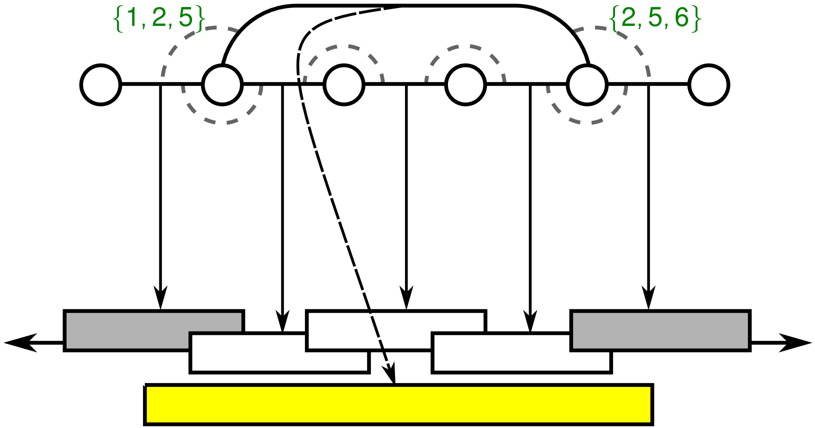

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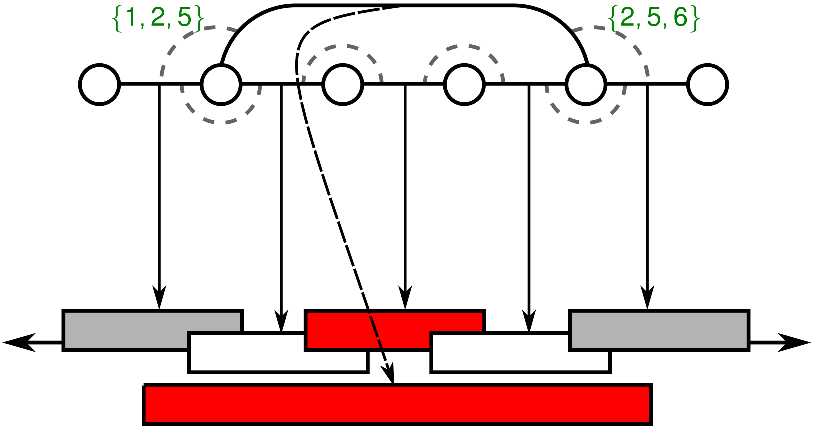

Ordered Ramsey Theory

An **ordered hypergraph** has a total order on the vertex set.

Let $G_1, \ldots, G_t$ be k-uniform ordered hypergraphs. Define the **ordered Ramsey number** $\mathrm{OR}(G_1, \ldots, G_t)$ to be the minimum N such that all t-colorings of $\binom{[n]}{k}$ contains an i-colored ordered copy of G_i for some $i \in [t]$. If $G_i = G$ for all $i \in [t]$, we write $\mathrm{OR}_t(G) = \mathrm{OR}(G, \ldots, G)$.

Since the complete k-uniform hypergraph K_n^k contains all ordered hypergraphs on n vertices, $\mathrm{OR}_t(K_n^k)$

Previous Work

Choudum and Ponnusamy (2002) defined **directed Ramsey theory** which involves coloring the acyclic tournament while avoiding monochromatic copies of **directed acyclic graphs**.

Their concept is different, but their results apply to 2-uniform ordered Ramsey theory.

Ordered Hyperpaths

Definition For $k \geq 2$ and $r \geq k$, the ***k*-uniform ordered path** P_r^k is the ordered graph on vertices $\{1, \ldots, r\}$ with edges $\{i, i+1, \ldots, i+k-1\}$ for all $i \in \{1, \ldots, r-k+1\}$.

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Uniformity: k.

Number of Vertices: r.

Number of Edges: $m = r - k + 1$.

Ordered Ramsey Numbers of Hyperpaths

Theorem (Folklore) $\mathrm{OR}(P^2_{r_1}, \ldots, P^2_{r_2}) = 1 + \prod_{i=1}^{t}(r_i - 1)$.

$\mathrm{OR}_t(P^2_r) = (r-1)^t + 1 = m^t + 1$.

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$\mathrm{OR}_t(P^2_r) = (r-1)^t + 1 = m^t + 1$.

In particular, $\mathrm{OR}_t(P^2_3) = 2^t + 1$.

In ordinary Ramsey theory, $\mathrm{R}_t(P_3) \leq t + 2$.

Ordered Ramsey Numbers of Hyperpaths

Theorem (Milans, Stolee, West, 2012+)

$$m^{\mathrm{tow}(k-2,t-O(\lg t))} \leq \mathrm{OR}_t(P_r^k) \leq \mathrm{tow}(k-1, t \lg m) + 1.$$

$$\mathrm{tow}(\ell, x) = \begin{cases} 2^{\mathrm{tow}(\ell-1,x)} & \ell \geq 1 \\ x & \ell = 0 \end{cases}.$$

Step-Up Lower Bound

Lemma (MSW12+) If $r > k \geq 2$, then $\mathrm{OR}_t(P_{r+1}^{k+1}) \geq \mathrm{OR}_{\binom{t}{\lfloor t/2 \rfloor}}(P_r^k)$.

Step-Up Lower Bound

Lemma (MSW12+) If $r > k \geq 2$, then $\mathrm{OR}_t(P_{r+1}^{k+1}) \geq \mathrm{OR}_{\binom{t}{\lfloor t/2 \rfloor}}(P_r^k)$.

Proof: Set $n = \mathrm{OR}_{\binom{t}{\lfloor t/2 \rfloor}}(P_r^k) - 1$.

Let $c : \binom{[n]}{k} \to \binom{[t]}{\lfloor t/2 \rfloor}$ be a coloring of K_n^k that avoids monochromatic copies of P_r^k.

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Define $c' : \binom{[n]}{k+1} \to [t]$ as follows for all $I \in \binom{[n]}{k+1}$ with

$$J = I - \max I, \qquad J' = I - \min I.$$

- If $c(J) = c(J')$, then select $c'(I) \in c(J)$.
- If $c(J) \neq c(J')$, then select $c'(I) \in c(J) - c(J')$.

Step-Up Lower Bound

If Q is the vertex set of an ordered copy of P_{r+1}^{k+1}, then $\hat{Q} = Q - \max Q$ is the vertex set of an ordered copy of P_r^k.

Some consecutive k-intervals $J, J' \subset \hat{Q}$ have distinct colors under c.

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$c'(I) \in c(J) - c(J')$ and $c'(I') \in c(J')$. Therefore, $c'(I) \neq c'(I')$, and the path on Q is not monochromatic.

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$$\mathrm{OR}_t(P_{r+1}^{k+1}) > n = \mathrm{OR}_{\binom{t}{\lfloor t/2 \rfloor}}(P_r^k) - 1.$$

Step-Up Upper Bounds

Lemma (MSW12+) If $k \geq 2$, then

- (Two Edges) $\mathrm{OR}_t(P^{k+1}_{k+2}) \leq \mathrm{OR}_{2^t}(P^k_{k+1})$, and
- (Three Edges) $\mathrm{OR}_t(P^{k+1}_{k+3}) \leq \mathrm{OR}_{2^{2t}}(P^k_{k+1})$.

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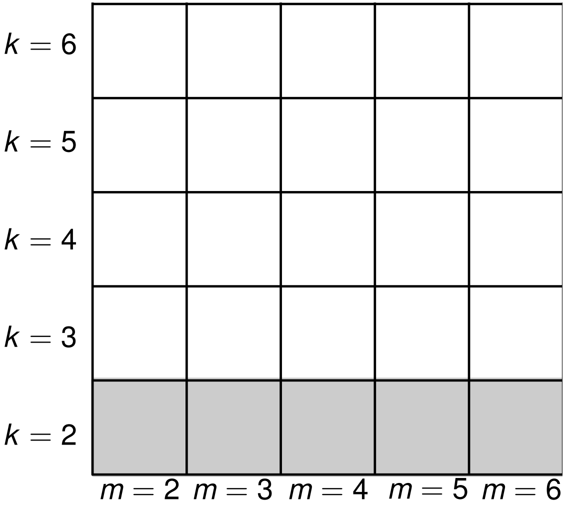
Lemma (MSW12+) If $r \geq k + 2 \geq 4$, then

$$\mathrm{OR}_t(P^{k+1}_r) \leq \mathrm{OR}_{(r-k)^t}(P^k_{k+1}).$$

Visual Guide to Step-ups

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$k=6$					
$k=5$					
$k=4$					
$k=3$					
$k=2$					

Visual Guide to Step-ups



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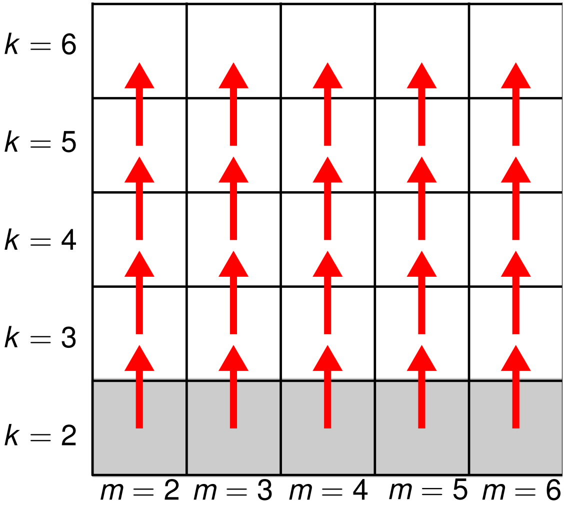

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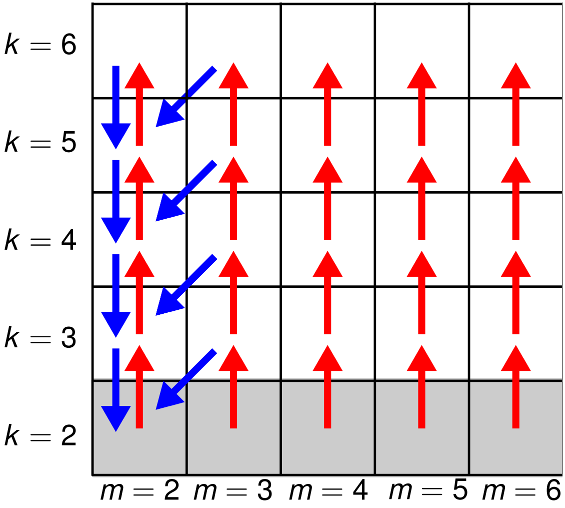

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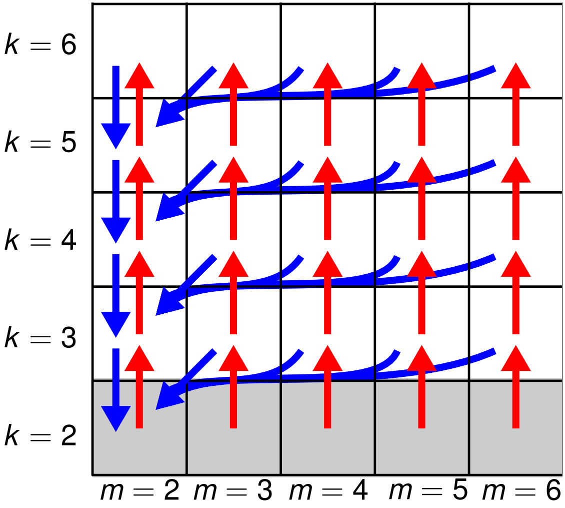

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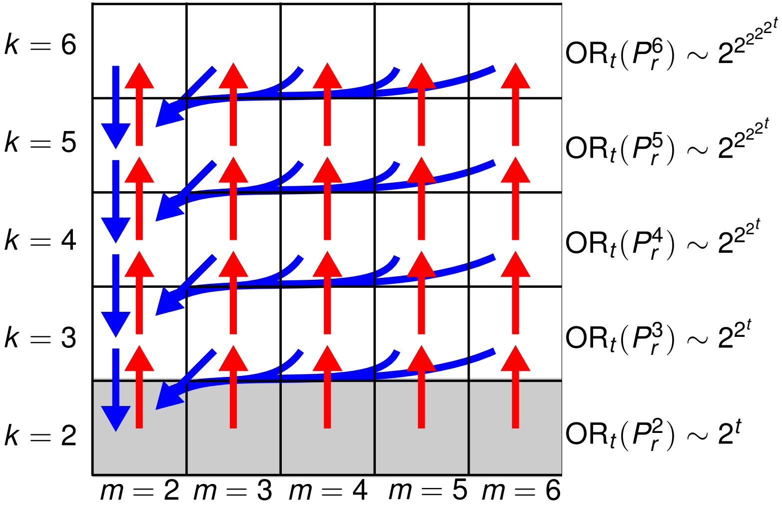
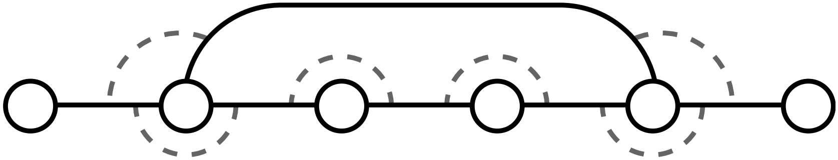


Asymptotics of $\tau(L(K_n))$

Theorem (Milans, Stolee, West, 2012+) If $\tau(L(K_n)) = t$, then

$$\mathrm{OR}_{t-3}(P_4^3) \leq n < \mathrm{OR}_t(P'),$$

where P' is the 3-uniform hypergraph formed from P_6^3 by adding the edges $\{1, 2, 5\}$ and $\{2, 5, 6\}$.



Asymptotics of $\tau(L(K_n))$

If $\tau(L(K_n)) = t$, then

$$2^{2^{t-O(\lg t)}} \leq \mathrm{OR}_{t-3}(P_4^3) \leq n < \mathrm{OR}_t(P') \leq 2^{2^{(4+o(1))t\lg t}}.$$

Theorem (Milans, Stolee, West, 2012+)

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Previous Work in Erdős-Szekeres Generalizations

Theorem (Fox, Pach, Sudakov, 2012)

$$2^{\frac{2}{3}m^{t-1}/\sqrt{t}} \leq \mathrm{OR}_t(P_r^3) \leq 2^{2m^{t-1}}.$$

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$$\mathrm{tow}(k-2, m^{t-1}/2\sqrt{t}) \leq \mathrm{OR}_t(P_r^k) \leq \mathrm{tow}(k-1, (t-1)\lg m).$$

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