Rainbow Arithmetic Progressions

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Ramsey Theory and anti-Ramsey Theory

Ramsey Theory: *Looking for monochromatic (monoχ) subgraphs in large edge-colored graphs.*

Anti-Ramsey Theory: *Looking for rainbow subgraphs in edge-colorings using many colors.*

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Ramsey Theory on the Integers

We will consider $[n] = \{1, \ldots, n\} \subset \mathbb{N}$. Let $k \geq 3$.

Definition A ***k*-term arithmetic progression (*k*-AP)** is a set S such that

$$S = \{a + id : 0 \leq i < k\} = \{a, a + d, a + 2d, \ldots, a + (k-1)d\}$$

for some integers a, d, and $d \neq 0$.

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Definition $\mathrm{w}_r(k)$ is the minimum n such that all r-colorings of $[n]$ contain a monoχ k-AP.

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Theorem (Behrend)

$$\mathrm{sz}([n], k) \geq n e^{-b\sqrt{\log n}}.$$

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Theorem (Juncić, Fox, Mahdian, Nešetřil, Radoičić) If $\mathbb{N}$ is colored with three colors such that each color class has upper density strictly larger than $\frac{1}{6}$, then the coloring contains a rainbow 3-AP.

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Theorem (Axenovich, Fon-Der-Flaass) If $[n]$ is colored with three colors such that each color class has size at least $\frac{n+4}{6}$, then the coloring contains a rainbow 3-AP.

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van der Waerden: *If the number of colors r is fixed and n is large, then there exist monoχ k-APs in every $c : [n] \to [r]$.*

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An r-coloring is **exact** if all colors are used at least once.

Definition $\mathrm{aw}([n], k)$ is the minimum positive r such that all exact r-colorings of $[n]$ contain a rainbow k-AP.

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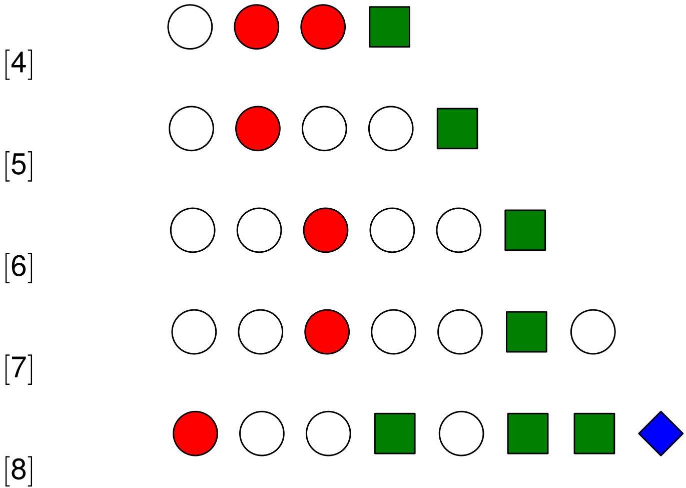
Assuming $k \leq n$:

$$k \leq \mathrm{aw}([n], k) \leq n$$

$n \setminus k$	3	4	5	6	7	8	9
3	3						
4	4						
5	4	5					
6	4	6					
7	4	6	7				
8	5	6	8				
9	4	7	8	9			
10	5	8	9	10			
11	5	8	9	10	11		
12	5	8	10	11	12		
13	5	8	11	11	12	13	
14	5	8	11	12	13	14	
15	5	9	11	13	14	14	15

Values of $\mathrm{aw}([n], k)$ for $3 \leq k \leq \frac{n+3}{2}$.

Extremal Colorings with no Rainbow 3-AP



Extremal Colorings with no Rainbow 3-AP

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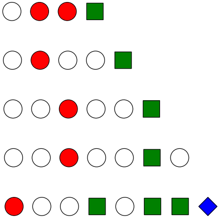

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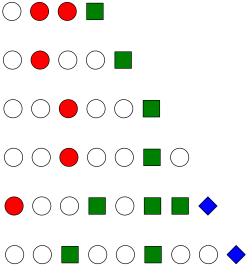

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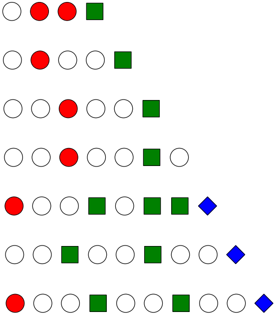

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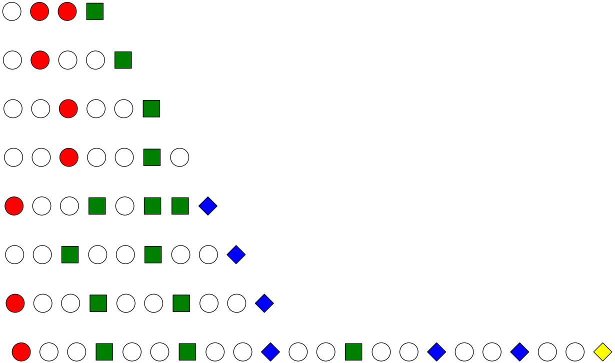

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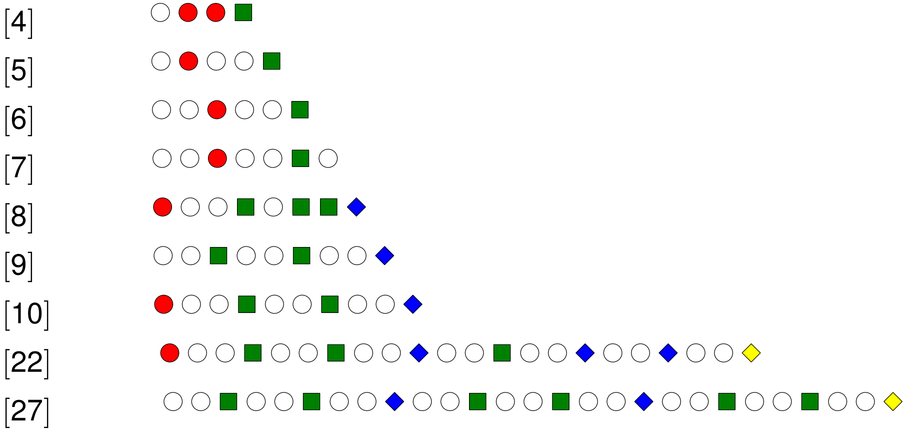

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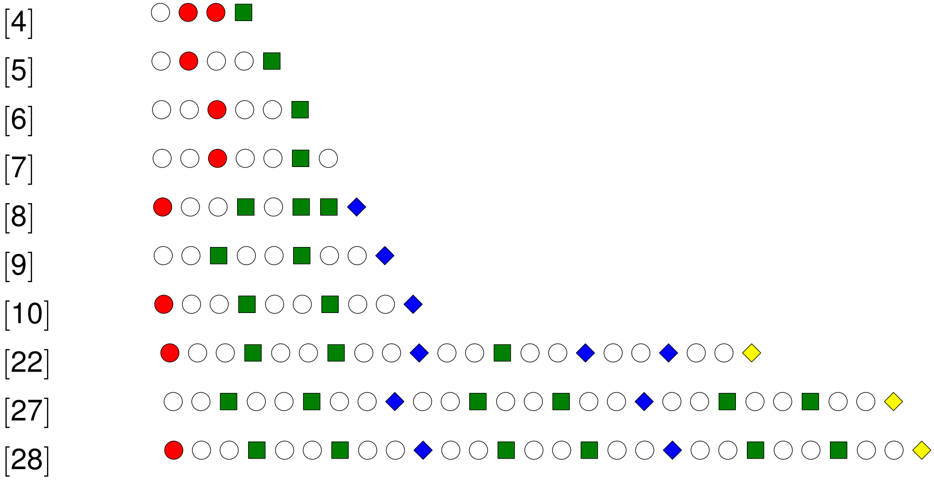

Monotonicity?

Proposition $\mathrm{aw}([n+1], k) \leq \mathrm{aw}([n], k) + 1$.

Conjecture $\mathrm{aw}([n+1], k) \geq \mathrm{aw}([n], k) - 1$.

Asymptotics of $\mathrm{aw}([n], k)$

Theorem (BEHHKKLMSWY '14)

$$\log_3 n + 2 \leq \mathrm{aw}([n], 3) \leq \log_2 n + 1$$

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$k = 3$ Case

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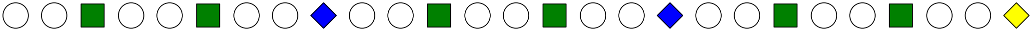
For the lower bound, consider the coloring

$$c(i) = \text{the largest integer } j \text{ such that } 3^j \text{ divides } i.$$



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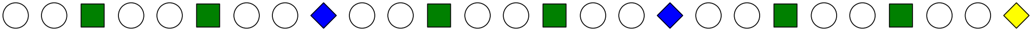

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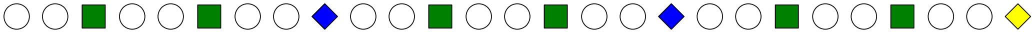

If $x < y < z$ is a 3-AP, then $x + z = 2y$. Let $j = \min\{c(x), c(y), c(z)\}$ and divide the equation by 3^j to find

$$m_x(3^{c(x)-j}) + m_z(3^{c(z)-j}) = 2m_y(3^{c(y)-j})$$

where m_x, m_y, m_z are relatively prime to 3.

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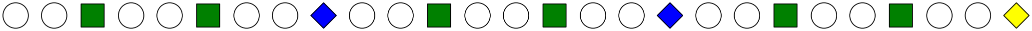

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where m_x, m_y, m_z are relatively prime to 3. Since the colors are distinct, exactly two of the numbers are multiples of three.

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For the $\log_2(n) + 1$ upper bound, consider the following

Proposition For $n \geq 2$, there exists $m \leq \lfloor \frac{n}{2} \rfloor$ such that

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There is a minimal interval $[a, b] \subset [n]$ such that all r colors appear. Thus, the color $c(a)$ does not appear within $[a + 1, b]$ and the color $c(b)$ does not appear within $[a, b - 1]$.

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Translate $[a, b]$ to be a coloring of $[t]$ where $t = b - a + 1$.

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Now, $c : [t] \to [r]$ is an exact coloring where $c(1) \neq c(t)$ and these colors do not appear within $[2, t-1]$.

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Claim: t is even. (If not, then $1, \frac{t-1}{2}, t$ is a rainbow 3-AP.)

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Now, $c : [t] \to [r]$ is an exact coloring where $c(1) \neq c(t)$ and these colors do not appear within $[2, t-1]$.

Claim: $r - 1$ colors appear on the odd elements of $[t]$.

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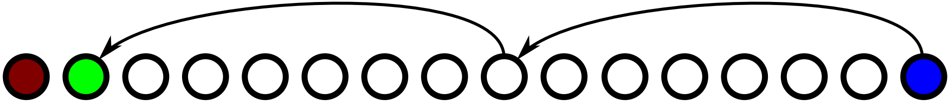

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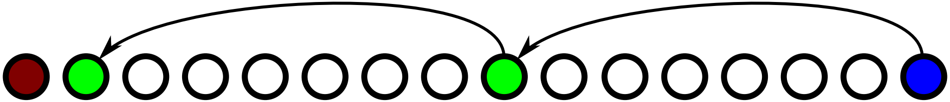

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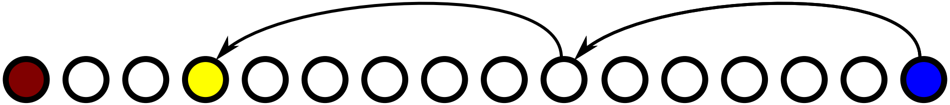

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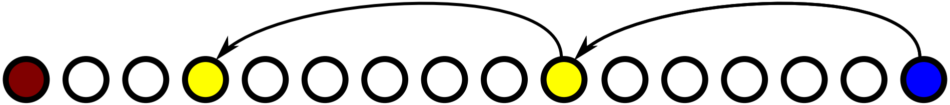

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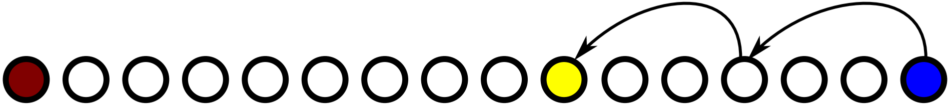

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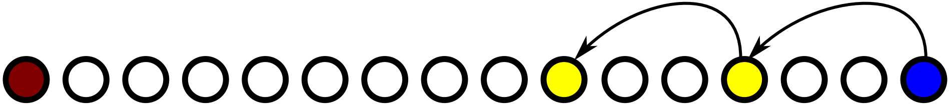

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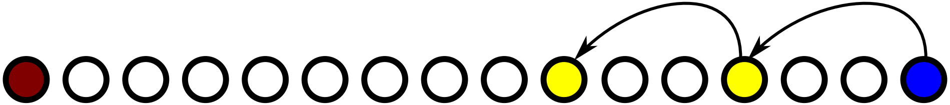
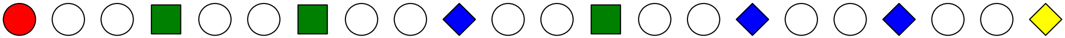
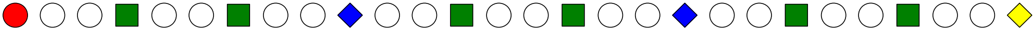


Note that $r - 1$ colors also appear on the even elements of $[t]$!

Structure of Extremal Colorings



$n = 22$



$n = 28$

The $k \geq 4$ Case

Theorem (BEHHKKLMSWY '14) For $k \geq 4$,

$$ne^{-O\left(\sqrt{\log n}\right)} < \mathrm{aw}([n], k) \leq ne^{-\log\log\log n - \omega(1)}.$$

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The upper bound follows from

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Let $r + 1 = \mathrm{aw}([n], k)$ and fix an exact r-coloring that avoids rainbow k-APs.

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Proof.

Let $r + 1 = \mathrm{aw}([n], k)$ and fix an exact r-coloring that avoids rainbow k-APs.

Select one element from each color class. This creates a set S of size r with no k-AP.

$$\mathrm{aw}([n], k) - 1 = |S| \leq \mathrm{sz}([n], k).$$

$\square$

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Proof. Let $S \subset [n]$ contain no $\lfloor k/2 \rfloor$-AP.

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Proof. Let $S \subset [n]$ contain no $\lfloor k/2 \rfloor$-AP. Color S with distinct colors and $[n] - S$ with another.

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Theorem (BEHHKKLMSWY '14) For $k \geq 4$,

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The lower bound follows somewhat from

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This is only non-trivial when $k \geq 6$.

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Let $S \subset [n]$ contain no punctured 4-AP. If we color S with distinct colors, then $[n] - S$ with a new color, the coloring avoids rainbow 4-APs.

anti-van der Waerden on $\mathbb{Z}_n$

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Assume $k \leq n$ and observe:

$$k \leq \mathrm{aw}(\mathbb{Z}_n, k) \leq \mathrm{aw}([n], k)$$

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	0	1	2	3	4	5	6	7	8	9
0–9				3	3	3	4	3	3	4
10–19	4	3	4	3	4	4	3	4	5	3
20–29	4	4	4	3	4	4	4	5	4	3
30–39	5	4	3	4	5	4	5	3	4	4
40–49	4	4	5	4	4	5	4	3	4	4
50–59	5	5	4	3	6	4	4	4	4	3
60–69	5	3	5	5	3	4	5	3	5	4
70–79	5	3	5	4	4	5	4	4	5	3
80–89	4	6	5	3	5	5	5	4	4	4
90–99	6	4	4	5	4	4	4	4	5	5

Computed values of $\mathrm{aw}(\mathbb{Z}_n, 3)$ for $n = 3, \ldots, 99$

The row label gives the range of n and the column heading is the ones digit within this range.

anti-van der Waerden on $\mathbb{Z}_n$, $k = 3$

Theorem (BEHHKKLMSWY '14)

1. For all positive integers m, $\mathrm{aw}(\mathbb{Z}_{2^m}, 3) = 3$.
2. For an integer $n \geq 2$ having every prime factor less than 100,

$$\mathrm{aw}(\mathbb{Z}_n, 3) = 2 + f_2 + f_3 + 2f_4.$$

Here f_4 denotes the number of odd prime factors of n in the set $Q_4 = \{17, 31, 41, 43, 73, 89, 97\}$. The quantity f_3 is the number of odd prime factors of n in Q_3, where Q_3 is the set of all odd primes less than 100 and not in Q_4. Both f_3 and f_4 are counted according to multiplicity. Finally, $f_2 = 0$ if n odd and $f_2 = 1$ if n is even.

anti-van der Waerden on $\mathbb{Z}_n$, $k \geq 4$

Theorem. For $k \geq 4$,

$$ne^{-O\left(\sqrt{\log n}\right)} < \mathrm{aw}(\mathbb{Z}_n, k) \leq ne^{-\log\log\log n - \omega(1)}.$$

Proof is essentially the same as the $\mathrm{aw}([n], k)$ case.

Open Problems

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Conjecture For positive integers n and k, $\mathrm{aw}([n], k) \geq \mathrm{aw}([n-1], k) - 1$.

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Question Is it true that $\mathrm{aw}([3n], 3) = \mathrm{aw}([n], 3) + 1$ for all positive integers n?

Open Problems

A **singleton extremal coloring** of S is an exact coloring of S that avoids rainbow k-APs and uses exactly $\mathrm{aw}(S, k) - 1$ colors

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Conjecture For p an odd prime and $t \geq 3$,

$$\mathrm{aw}(\mathbb{Z}_{pt}, 3) \geq \mathrm{aw}(\mathbb{Z}_t, 3) + \mathrm{aw}(\mathbb{Z}_p, 3) - 2.$$

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Question Does there exist a prime p such that $\mathrm{aw}(\mathbb{Z}_p, 3) \geq 5$?

Rainbow Arithmetic Progressions II: The Collaboration

Steve Butler   Craig Erickson   Leslie Hogben
Kirsten Hogenson   Lucas Kramer   Richard L. Kramer
Jephian C.-H. Lin   Ryan R. Martin   Derrick Stolee*
Nathan Warnberg   Michael Young

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May 19th, 2014
University of Colorado Denver

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People take turns at board with ideas, and taking suggestions from group.

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A new problem.

What Didn’t Work

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Ideal: no more than 4 grad students and 2 faculty per group.

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An 11-author paper will look confusing on anyone’s C.V.

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Group Rotation: Have one group meet in seminar room per week. Other groups meet students-only in another room.

Rainbow Arithmetic Progressions

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