Automated Discharging Arguments for Density Problems in Grids

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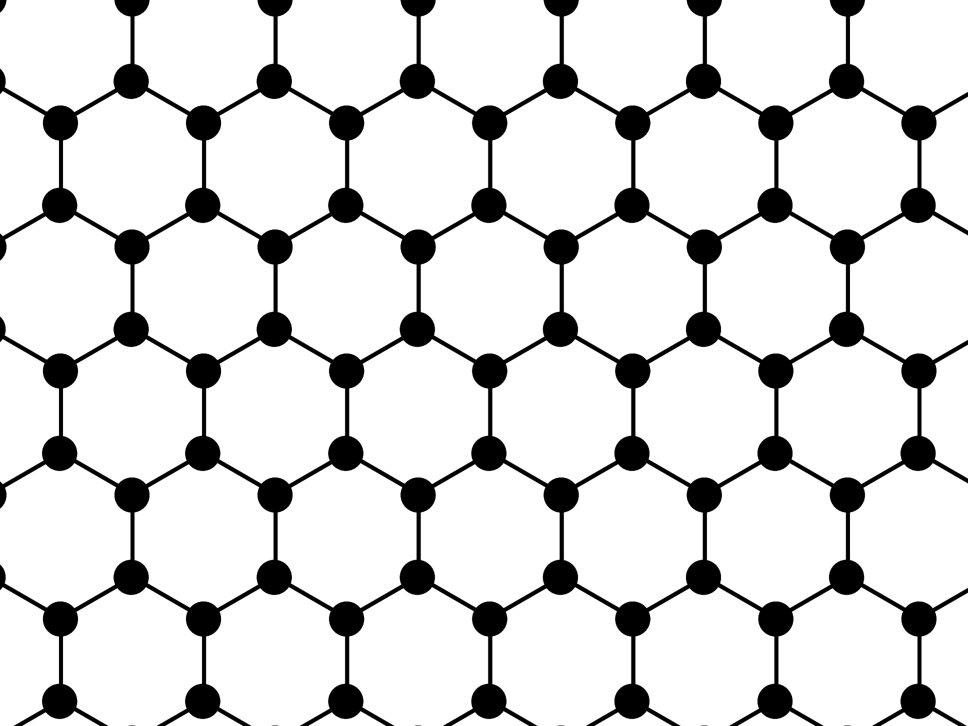
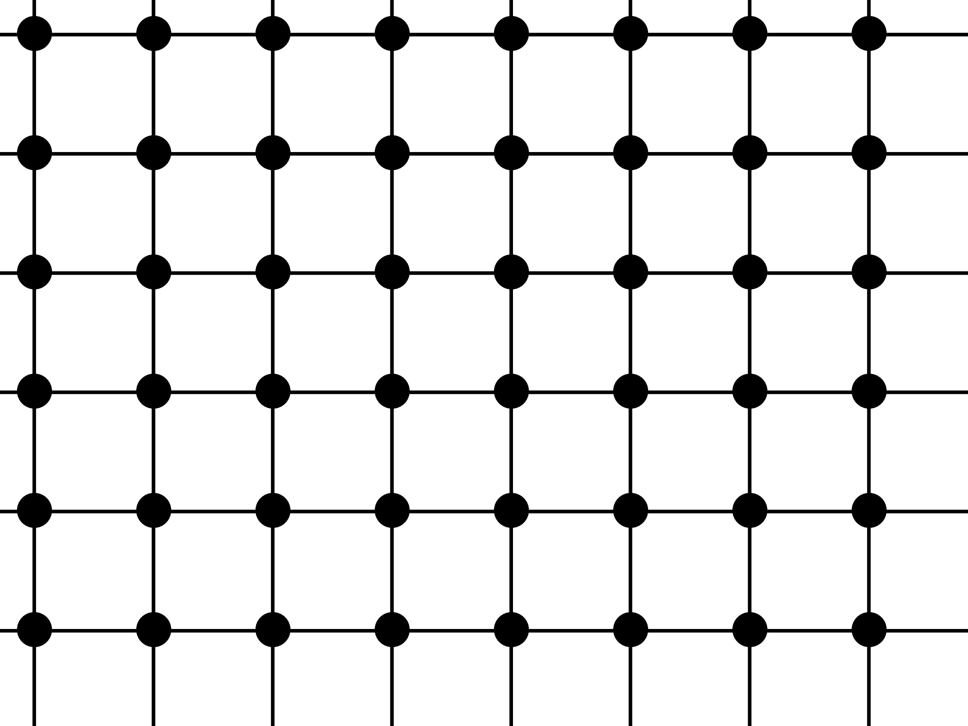
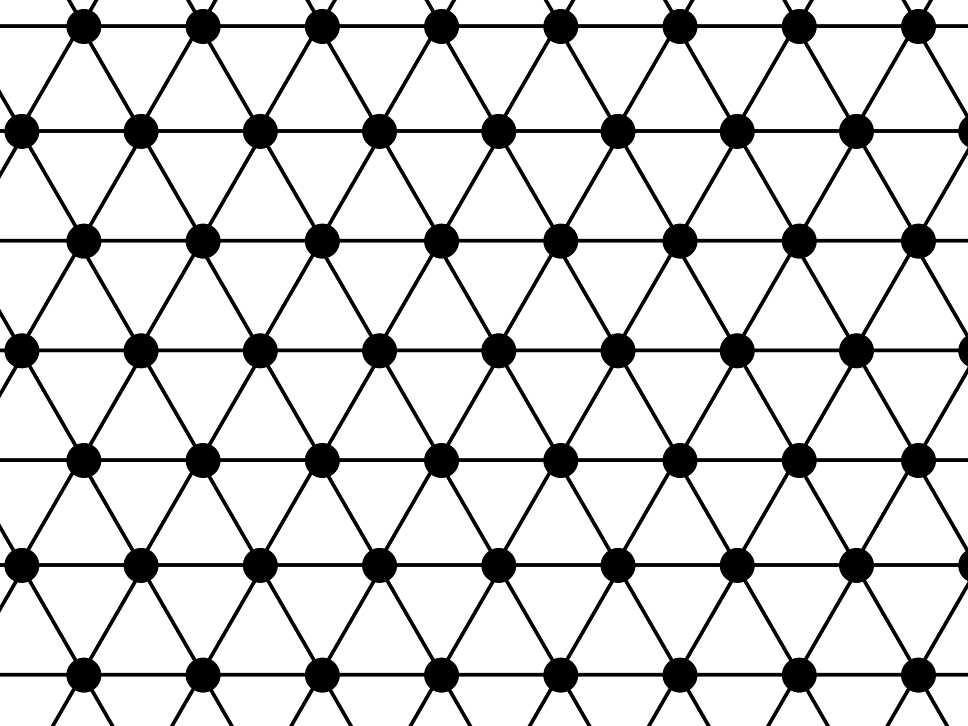
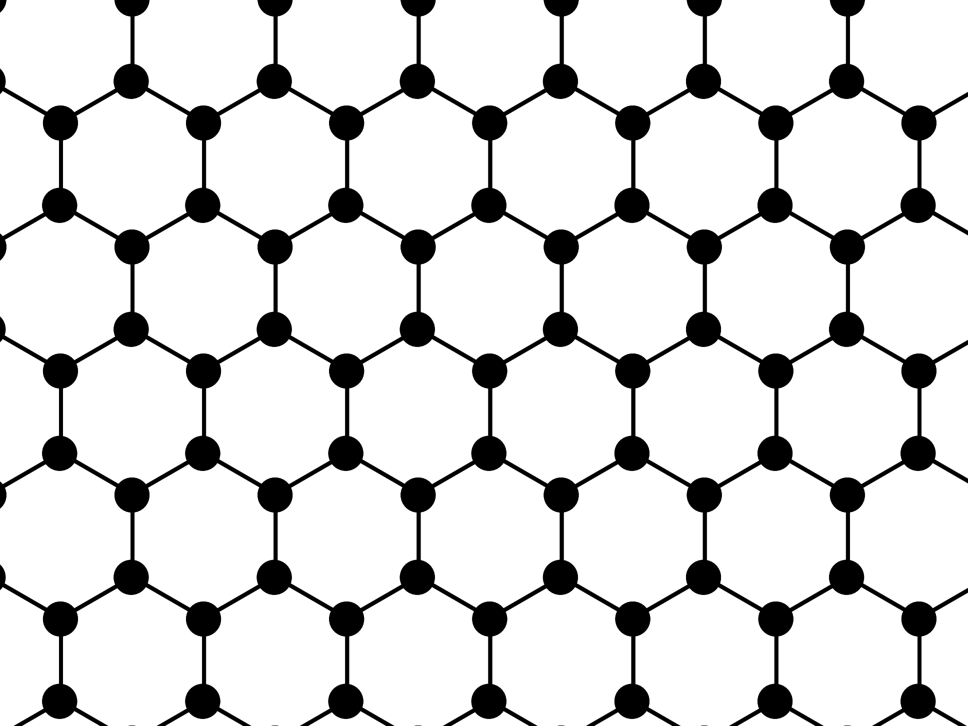
Ron Graham Conference









Density

Therefore, we can select an arbitrary vertex $v_0 \in V(G)$ and define the **density** of a set $X \subseteq V(G)$ as

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This definition is used for problems where we **minimize** the density.

We would use $\liminf$ for maximizing the density.

Dominating Sets

A set $X \subseteq V(G)$ is a **dominating set** if

- $N[v] \cap X \neq \varnothing$ for all vertices $v \in V(G)$.

($N[v]$ is the **closed neighborhood** of v: $N[v] = N(v) \cup \{v\}$.)

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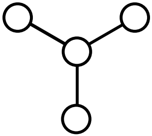

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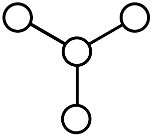

Forbidden Configuration

It is not difficult to see that the optimal density of a dominating set in the hexagonal grid is $\frac{1}{4} = 0.250000$.

Identifying Codes

A set $X \subseteq V(G)$ is an **identifying code** if

- $N[v] \cap X \neq \varnothing$ for all vertices $v \in V(G)$, and
- $N[v] \cap X \neq N[u] \cap X$ for all distinct vertices $v, u \in V(G)$.

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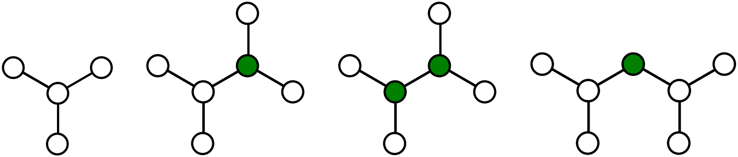

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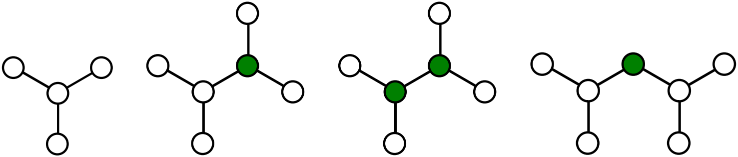

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Defined by Karpovsky, Chakrabarty, Levitin in 1998.

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Let G be the hexagonal grid, and

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Discharging demonstrates a connection between **local structure** and **global averages**.

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By **discharging**, we aim to distribute the charge evenly.

If the final charge amount is bounded below by the same value, then we have a bound on the **global average**.

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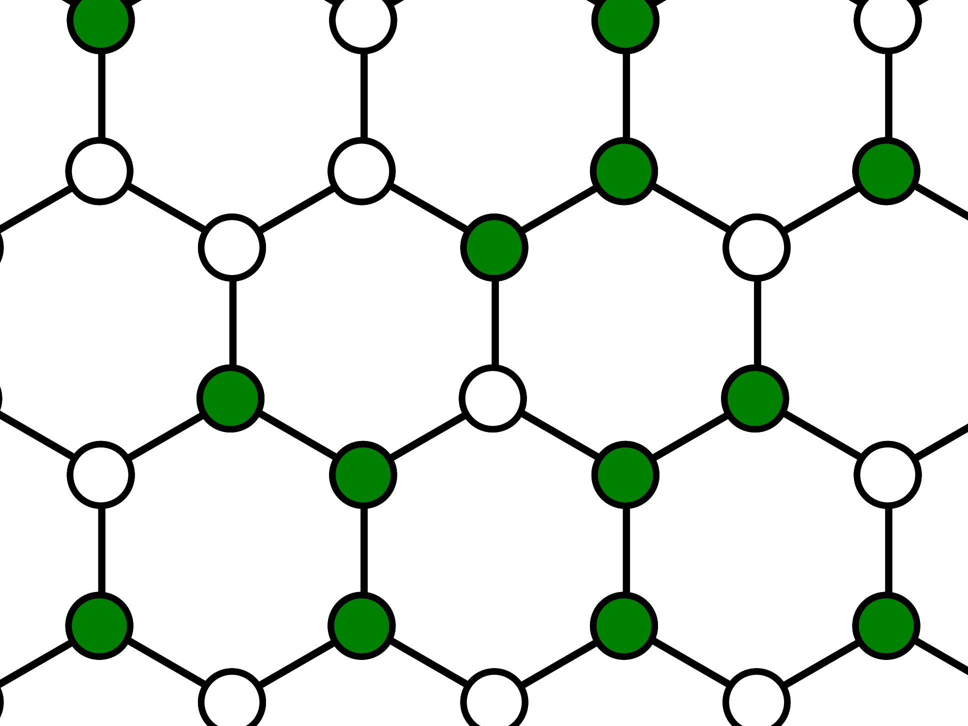
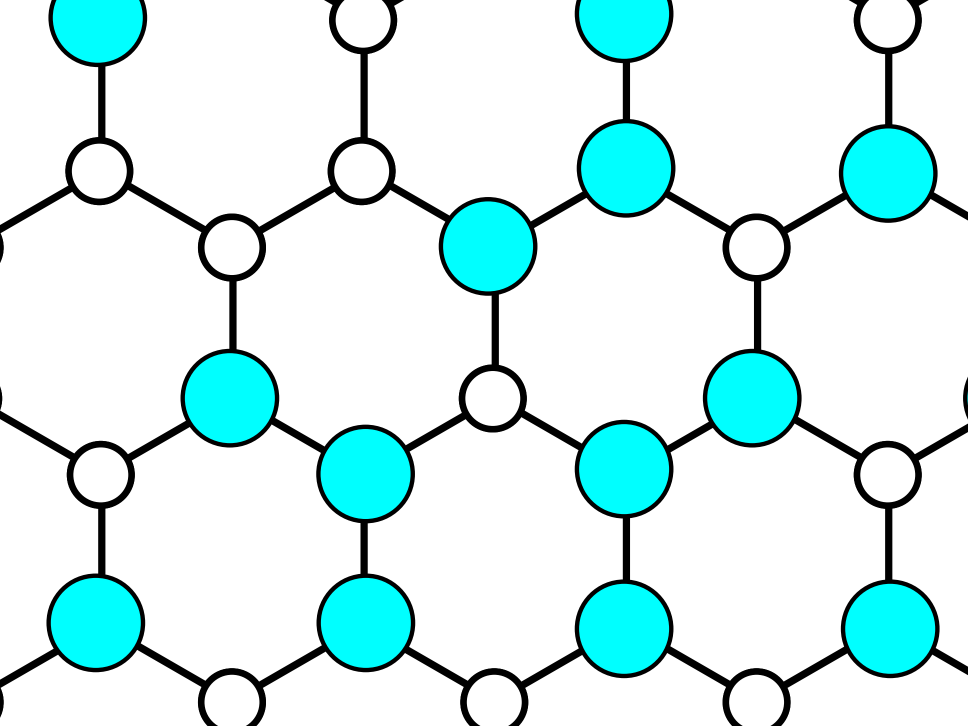
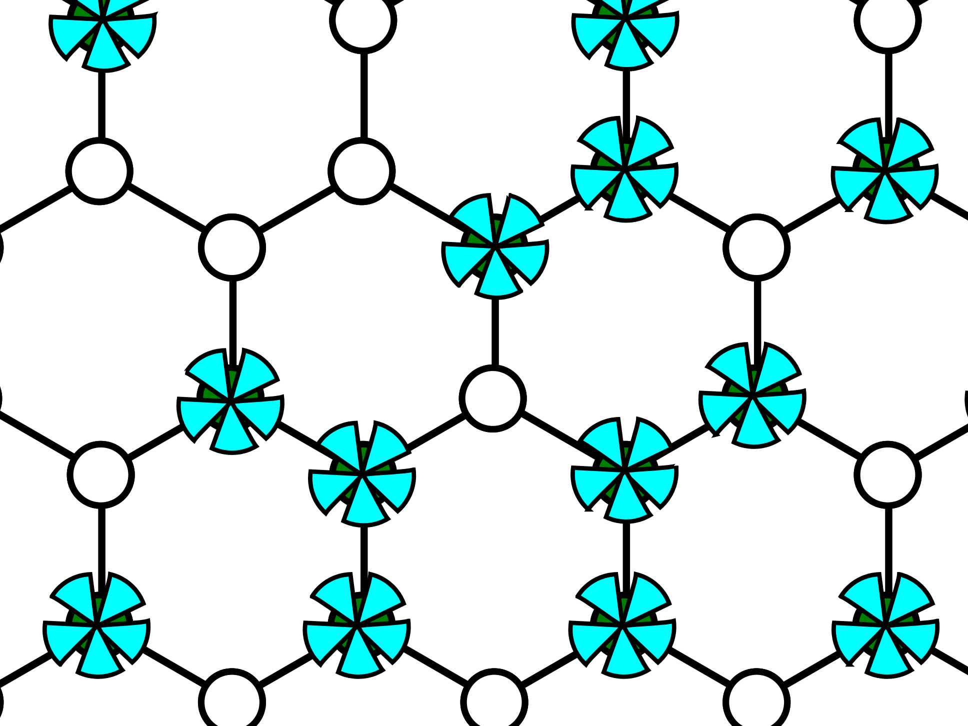
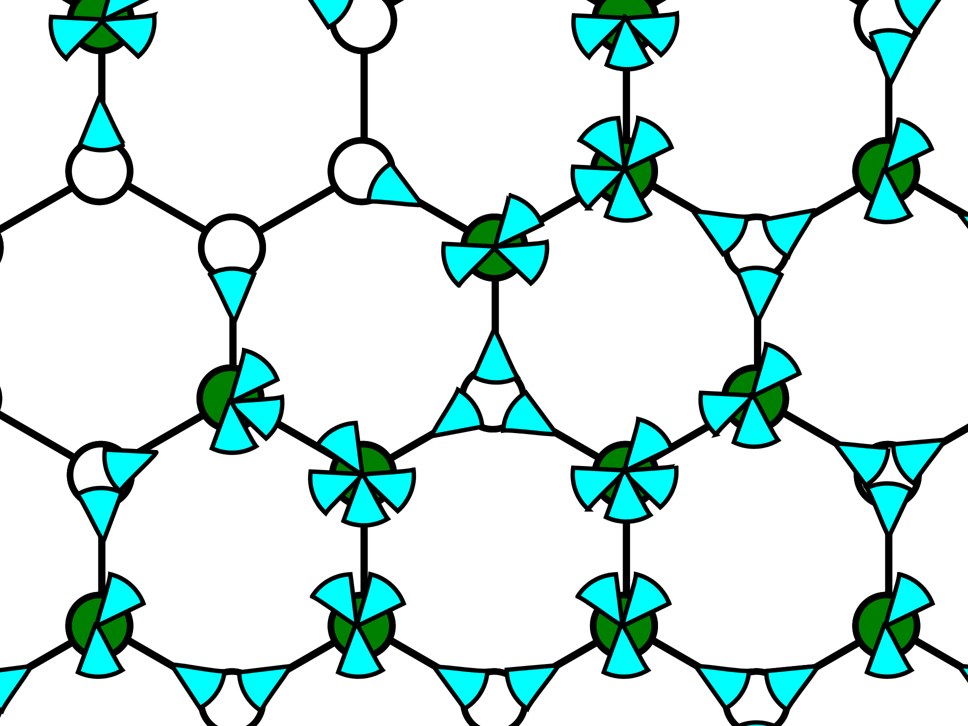
If we **discharge** such that our new charge values $\mu'(v)$ have $\mu'(v) \geq w$ always, then

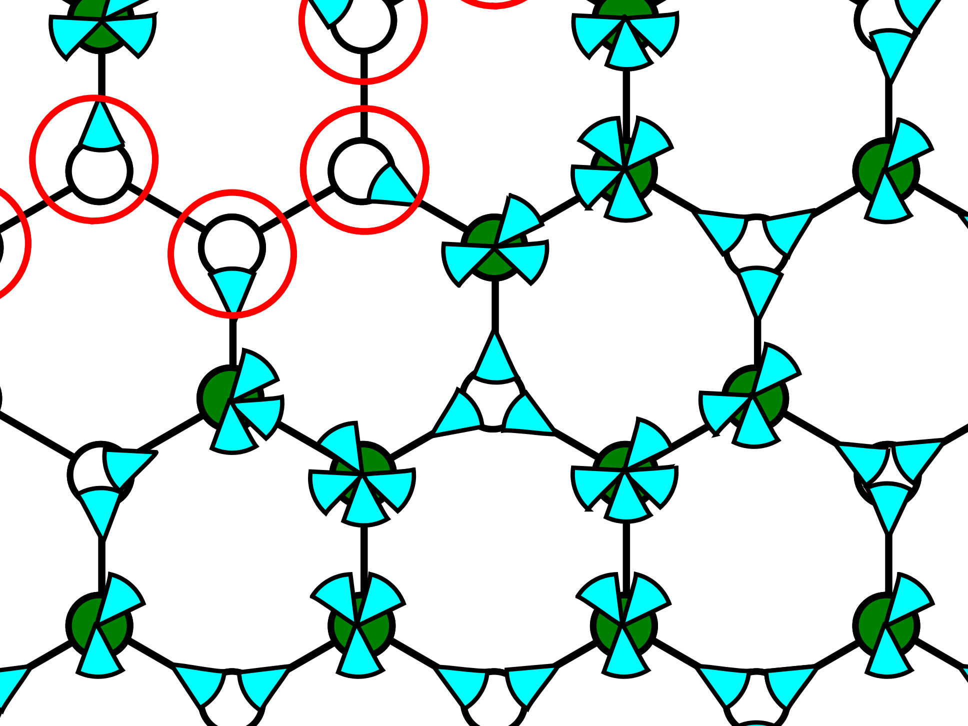
$$\delta(X) = \limsup_{r\to\infty} \frac{\sum_{v \in B_r(v_0)} \mu(v)}{|B_r(v_0)|} = \limsup_{r\to\infty} \frac{\sum_{v \in B_r(v_0)} \mu'(v)}{|B_r(v_0)|} \geq w.$$

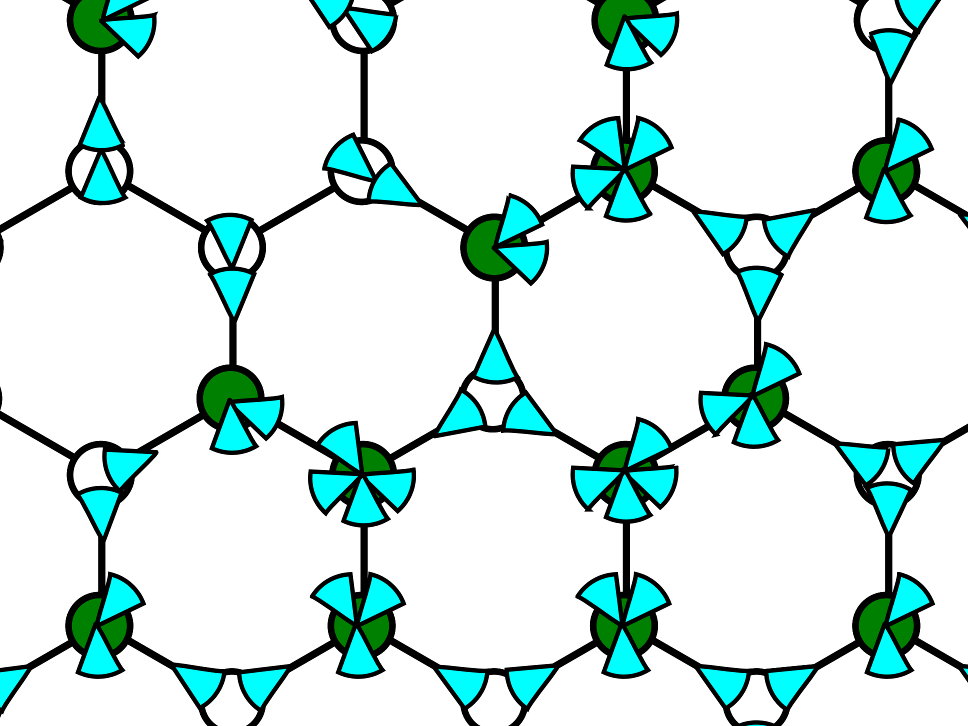












Example Discharging Argument

(R1) Every element sends charge $\frac{1}{5}$ to every adjacent nonelement.

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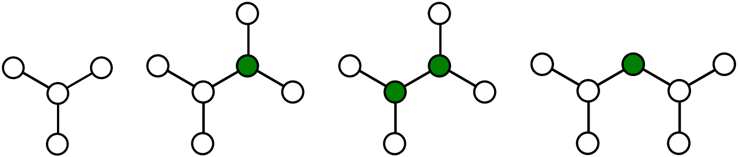
Why does it work?



Forbidden Configurations

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The equality

$$\limsup_{r\to\infty} \frac{\sum_{v\in B_r(v_0)} \mu(v)}{|B_r(v_0)|} = \limsup_{r\to\infty} \frac{\sum_{v\in B_r(v_0)} \mu'(v)}{|B_r(v_0)|}$$

holds only when our discharging sends a **bounded amount** of charge a **bounded distance**.

Discharging Arguments

The main difficulty with designing discharging arguments is to balance

- Low-charge objects *receive* enough charge to match the goal value.
- High-charge objects *maintain* enough charge to match the goal value.

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ADAGE

A proof using this technique is called an **adage**.

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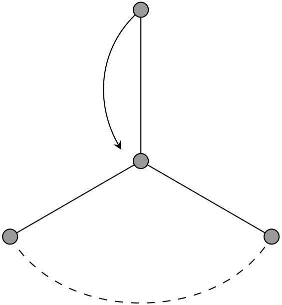
3. **Optimize** the values.

Defining a Rule



center object

chargeable objects

keys

kernel

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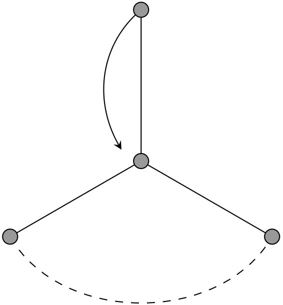

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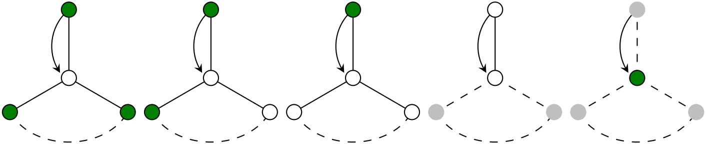

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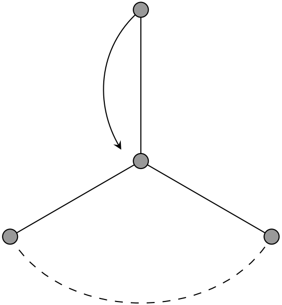

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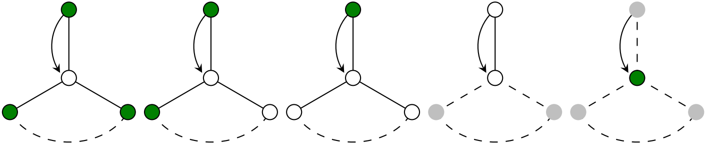
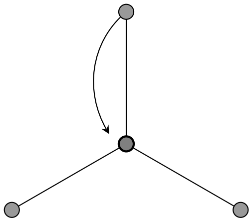
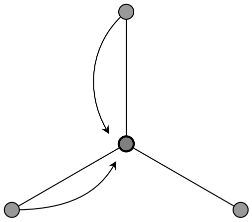
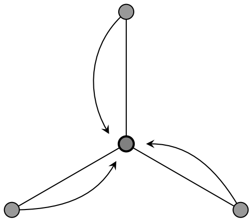
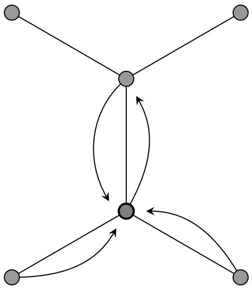


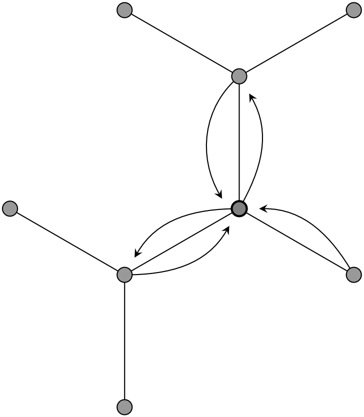
`s(0,1,2) s(0,1,1) s(0,1,0) s(0,0,_) s(1,_,_)`

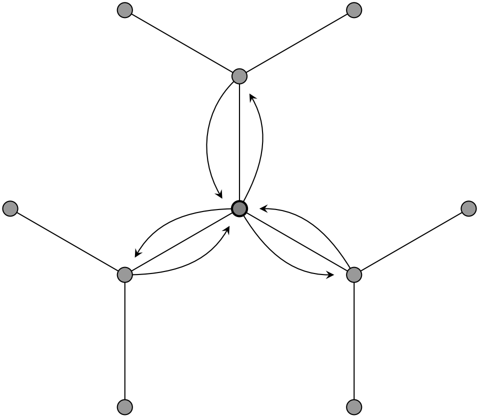


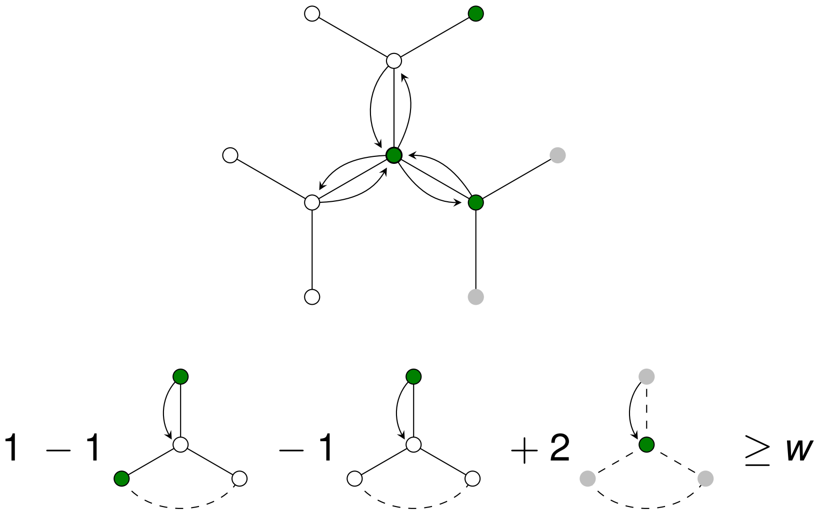




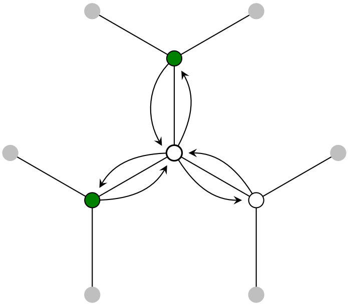


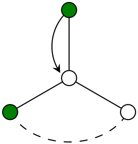
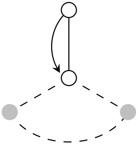
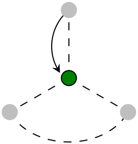


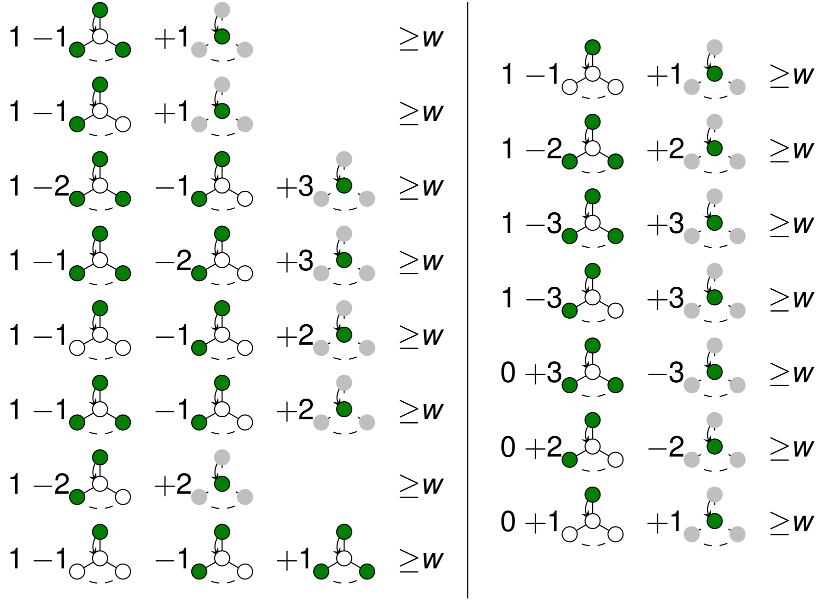




1 − 1
− 1
+ 2
≥ w



$0 + 2$  $+ 0$  $- 2$  $\geq w$



1 −1 +1 ≥w
1 −1 +1 ≥w
1 −2 −1 +3 ≥w
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1 −2 +2 ≥w
1 −1 −1 +1 ≥w
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1 −2 +2 ≥w
1 −3 +3 ≥w
1 −3 +3 ≥w
0 +3 −3 ≥w
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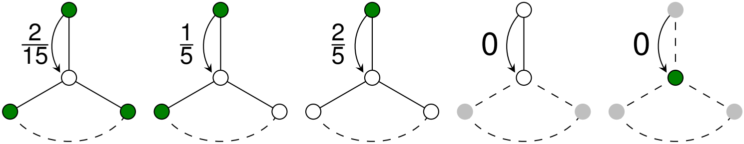
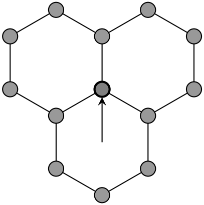
0 +1 +1 ≥w

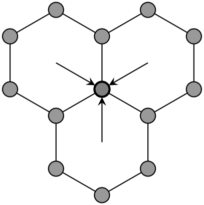


$$w = \frac{2}{5} = 0.40000$$

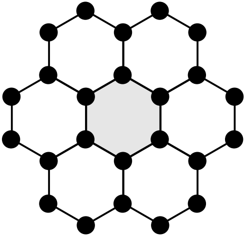
Results



Rule *N*
1,758 Realizations



Vertex
1,758 Constraints

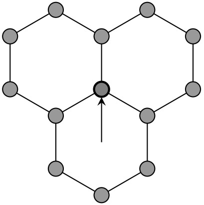


Face
663,662 Constraints

Theorem

Let X be an identifying code in the hexagonal grid. The adage proof using rule N demonstrates a lower bound of $\delta(X) \geq \frac{23}{55} - \frac{1}{496496} \approx 0.418181$.

Results



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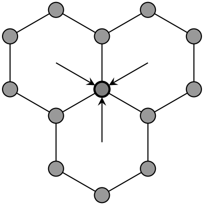
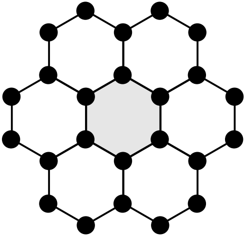


Vertex
1,758 Constraints



Face
663,662 Constraints

Theorem

Let X be an identifying code in the hexagonal grid. The adage proof using rule N demonstrates a lower bound of $\delta(X) \geq \frac{23}{55} - \frac{1}{496496} \approx 0.418181$.

This improves the previous-best lower bound of Cuickerman & Yu ($\frac{5}{12} = 0.41\overline{6}$) but does not match the current-best upper bound ($\frac{3}{7} \approx 0.42857$).

Results for Variations on Identifying Codes

Set Type	**Hexagonal Grid**		**Square Grid**		**Triangular Grid**	
Dominating Set	V_1	$\frac{1}{4} \approx 0.250000$*	V_1	$\frac{1}{5} \approx 0.200000$*	V_1	$\frac{1}{7} \approx 0.142857$*
Identifying Code	N	$\frac{23}{55} \approx 0.418182^{\dagger}$	V_2	$\frac{7}{20} \approx 0.350000$*	V_1	$\frac{1}{4} \approx 0.250000$*
Strong Identifying Code	V_2	$\frac{8}{17} \approx 0.470588$	$C_1 \cup C_2$	$\frac{7}{18} \approx 0.388889$	$C_1^+ \cup C_2^+$	$\frac{4}{13} \approx 0.307692$
Locating-Dominating Code	V_2	$\frac{1}{3} \approx 0.333333$*	V_2	$\frac{3}{10} \approx 0.300000$*	$C_1 \cup C_2$	$\frac{12}{53} \approx 0.226415$
Open-Locating-Dominating Code	V_2	$\frac{1}{2} \approx 0.500000$*	C_1^+	$\frac{2}{5} \approx 0.400000$*	C_1^+	$\frac{4}{13} \approx 0.307692$*

Automated Discharging Arguments for Density Problems in Grids

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