$(4, 2)$-Choosability of Planar Graphs with Forbidden Structures
A Working Seminar Report

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ISU Discrete Math Seminar

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Working Seminar 2014–15

Coloring and Choosing

Colorings and List Colorings

A **(proper) k-coloring** of a graph is an assignment $c : V(G) \to \{1, \ldots, k\}$ where

$$c(u) \neq c(v) \text{ for all } uv \in E(G).$$

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$$c(u) \neq c(v) \text{ for all } uv \in E(G).$$

A **list assignment** is a function $L : V(G) \to 2^{\mathbb{N}}$. An **$L$-coloring** is an assignment $c : V(G) \to \mathbb{N}$ such that

$$c(v) \in L(v) \text{ and } c(u) \neq c(v) \text{ for all } uv \in E(G).$$

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A graph is ***k*-colorable** if a k-coloring exists for G.

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A graph can be k-colorable but not k-choosable.

A (k, c)**-list assignment** is a list assignment L where

- $|L(v)| \geq k$ for all $v \in V(G)$
- $|L(u) \cap L(v)| \leq c$ for all $uv \in E(G)$.

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Let $f : V(G) \to \mathbb{N}$ be a funciton. A graph is (f, c)**-choosable** if it is L-colorable for every list assignment L where

- $|L(v)| \geq f(v)$ for all $v \in V(G)$
- $|L(u) \cap L(v)| \leq c$ for all $uv \in E(G)$.

$(3, 1)$-choosability

Conjecture (Škrekovski) If G is a planar graph, then G is $(3, 1)$-choosable.

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Theorem. Let G be a planar graph. G is $(3,1)$-choosable if G avoids any of the following structures:

- 3-cycles (Kratochvíl, Tuza, Voigt, Choi, Lidický, Stolee).
- 4-cycles (Choi, Lidický, Stolee).
- 5-cycles and 6-cycles (Choi, Lidický, Stolee).

4-choosability

Theorem. Let G be a planar graph. G is 4-choosable if G avoids any of the following structures:

- 3-cycles (Folklore).
- 4-cycles (Lam, Xu, Liu).
- 5-cycles (Wang and Lih).
- 6-cycles (Fijavz, Juvan, Mohar, and Škrekovski).
- 7-cycles (Farzad).
- Chorded 4-cycles and chorded 5-cycles (Borodin and Ivanova).

$(4, c)$-choosability

Theorem (Kratochvíl, Tuza, and Voigt) If G is a planar graph, then G is $(4, 1)$-choosable.

Theorem (Voigt) There exists a planar graph that is not $(4, 3)$-choosable.

$(4,2)$-choosability

A **chorded k-cycle** is a k-cycle with one additional edge.

A **doubly-chorded k-cycle** is a k-cycle with two additional edges.

$(4, 2)$-choosability

A **chorded k-cycle** is a k-cycle with one additional edge.

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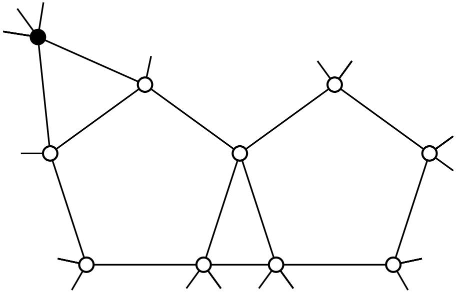
Theorem

Let G be a planar graph. G is $(4, 2)$-choosable if G avoids any of the following structures:

- *Chorded 5-cycles.*
- *Chorded 6-cycles.*
- *Chorded 7-cycles.*
- *Doubly-chorded 6-cycles and doubly-chorded 7-cycles.*

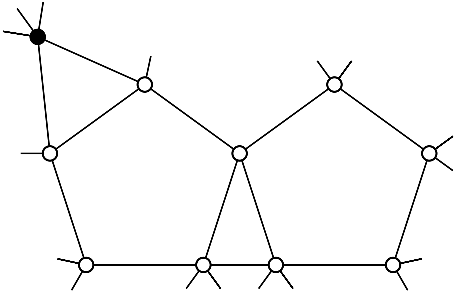
Reducible Configurations

Configuration



- Graph C
- Set $X \subseteq V(C)$
- Function $\mathrm{ex} : V(C) \to \{0, 1, 2, \infty\}$
- Function $f : X \to \{1, 2, 3, 4\}$ (roughly $f(v) = 4 - \mathrm{ex}(v)$)

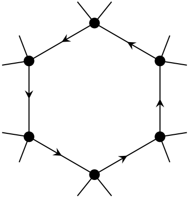
Configuration



A configuration is **reducible** if C is $(f,2)$-choosable.

The Alon-Tarsi Theorem

A digraph D is an *orientation* of G if G is the underlying undirected graph of D and D has no 2-cycles.



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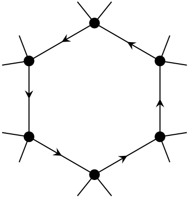

An *Eulerian subgraph* of a digraph D is a subset $S \subseteq E(D)$ such that $d_S^+(v) = d_S^-(v)$.

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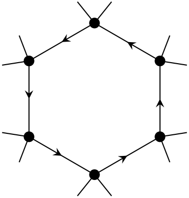

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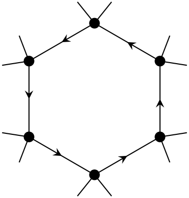
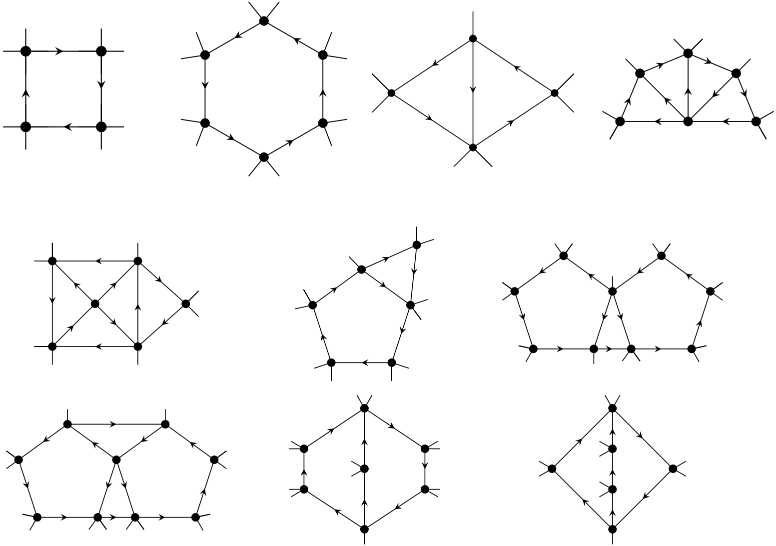
- $EE(D)$: number of Eulerian subgraphs of even size.
- $EO(D)$: number of Eulerian subgraphs of odd size.

The Alon-Tarsi Theorem

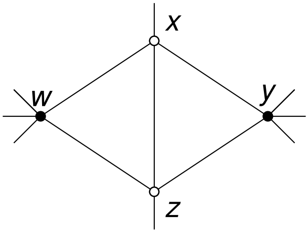
Theorem (Alon-Tarsi) Let G be a graph and $f : V(G) \to \mathbb{N}$ a function. Suppose there exists an orientation D of G such that $d_D^+(v) \leq f(v) - 1$ for every vertex $v \in V(G)$ and $EE(D) \neq EO(D)$. Then G is f-choosable.



Using the Alon-Tarsi Theorem



A $(4, 2)$-Reducibility Example



$f(x) = f(z) = 3$, $f(w) = f(y) = 1$

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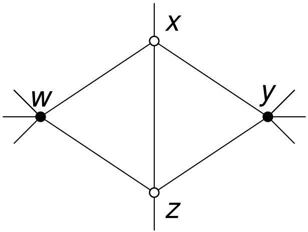

$f(x) = f(z) = 3$, $f(w) = f(y) = 1$

Let $L'(x) = L(x) \setminus (L(y) \cup L(w))$ and
$L'(z) = L(z) \setminus (L(y) \cup L(w))$.

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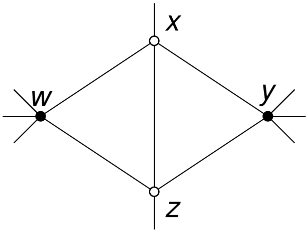
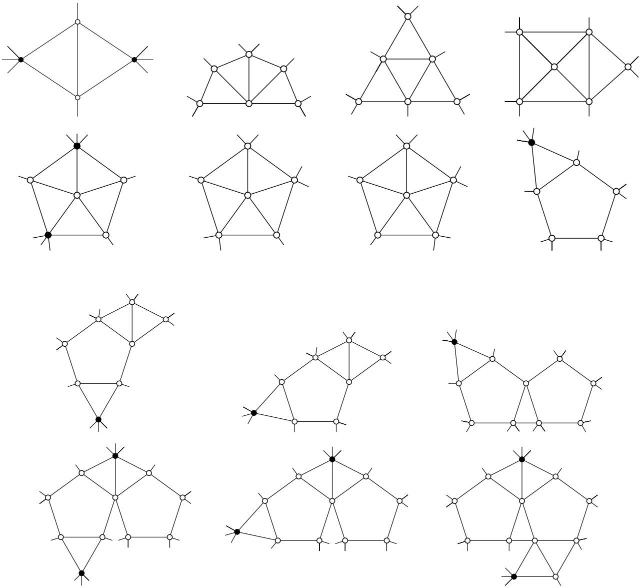


$f(x) = f(z) = 3$, $f(w) = f(y) = 1$

Let $L'(x) = L(x) \setminus (L(y) \cup L(w))$ and
$L'(z) = L(z) \setminus (L(y) \cup L(w))$.

If $|L'(x)| = |L'(z)|$, then $L(x) \cap L(z) = L(w) \cup L(y)$ and hence
$L'(x) \cap L'(z) = \emptyset$.

Other $(4, 2)$-Reducible Configurations



Discharging

Theorem. If G is a planar graph not containing a chorded 6-cycle, then G is $(4, 2)$-choosable.

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Suppose there exists a **counterexample**. Select a plane graph G that:

- Does not contain a chorded 6-cycle.
- Has a $(4, 2)$-list assignment L where G is not L-choosable.
- $n(G)$ is as small as possible.

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A 5-vertex is incident to at most **three** 3-faces.

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G does not **contain** a reducible configuration.

Initial Charge Functions

For a vertex v, let $\mu(v) = d(v) - 4$.

For a face f, let $\nu(f) = \ell(f) - 4$.

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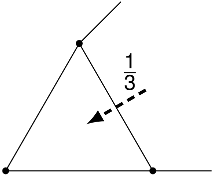
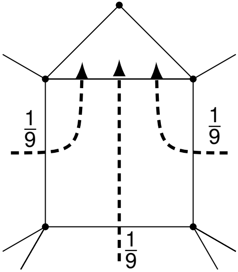
Goal: Move charge around so everything has nonnegative charge, a contradiction!

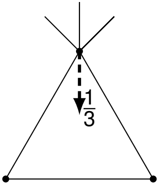
Discharging Rules



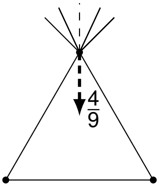
(R1a)



(R1b)



(R2a)



(R2b)

Verifying Discharging Rules

Case 1: Let f be a 6^+-face ($\ell(f) \geq 6$).

$$\nu'(f) \geq \underbrace{\ell(f) - 4}_{\text{initial charge}} - \underbrace{\frac{1}{3}\ell(f)}_{\text{amount lost by (R1)}} = \frac{2}{3}\ell(f) - 4 \geq 0.$$

Verifying Discharging Rules

Case 2: Let f be a 5-face ($\ell(f) = 5$).

$$\nu'(f) \geq \underbrace{\ell(f) - 4}_{\text{initial charge}} - \underbrace{\frac{1}{9}\ell(f)}_{\text{amount lost by (R1)}} = \frac{8}{9} \cdot \ell(f) - 4 \geq 0.$$

(No 5-face adjacent to a 3-face.)

Verifying Discharging Rules

Case 3: Let f be a 4-face ($\ell(f) = 4$).

$\nu(f) = 0$ and f does not gain or lose charge by (R1).

(No 4-face adjacent to a 4-face.)

Verifying Discharging Rules

Case 4: Let v be a 6^+-vertex.

$$\mu'(v) \geq \underbrace{d(v) - 4}_{\text{initial charge}} - \underbrace{\frac{4}{9}\left\lfloor \frac{3d(v)}{4} \right\rfloor}_{\text{amount lost by (R2)}} \geq \frac{2}{3}d(v) - 4 \geq 0.$$

(A 6^+-vertex is incident to at most $\left\lfloor \frac{3}{4}d(v) \right\rfloor$ 3-faces.)

Verifying Discharging Rules

Case 5: Let v be a 5-vertex.

$$\mu'(v) \geq \underbrace{d(v) - 4}_{\text{initial charge}} - \underbrace{\frac{1}{3} \cdot 3}_{\text{amount lost by (R2)}} \geq 0.$$

(A 5-vertex is incident to at most three 3-faces.)

Verifying Discharging Rules

Case 6: Let v be a 4-vertex.

$\mu(v) = 0$ and v does not gain or lose charge via (R2).

Verifying Discharging Rules

Case 7: Let f be a 3-face.

$\nu(f) = -1$.

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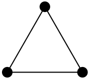
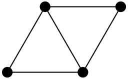
Not every 3-face will gain enough charge!

But we will average among 3-faces in each **cluster**.

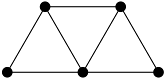
Clusters



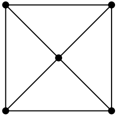
(K3)



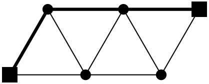
(K4)



(K5a)



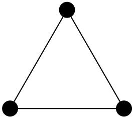
(K5b)



(K5c)

(These are the maximal connected sets of 3-faces that do not contain a 3^--vertex or a chorded 6-cycle.)

Cluster (K3)

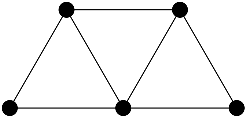


Initial Charge: -1

Gained by (R1): $3 \cdot \frac{1}{3}$

Charge after (R1): 0

Cluster (K5a)

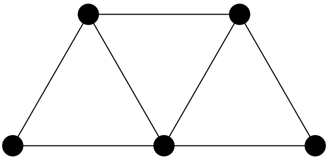


Initial Charge: -3

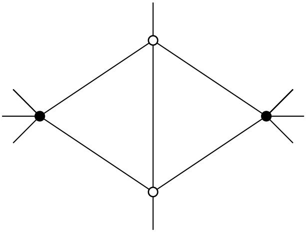
Gained by (R1): $5 \cdot \frac{1}{3}$

Charge after (R1): $-\frac{4}{3}$

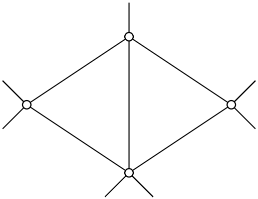
Cluster (K5a)



After (R1): $-\frac{4}{3}$

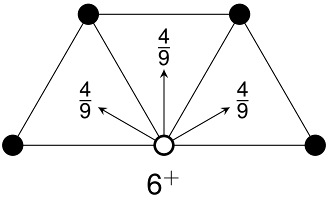


(C3)



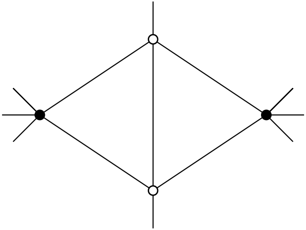
(C4)

Cluster (K5a)



After (R1): $-\frac{4}{3}$

After (R2): $-\frac{4}{3} + 3 \cdot \frac{4}{9} = 0$



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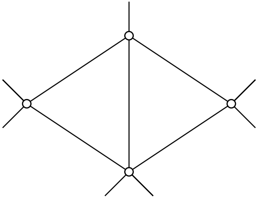

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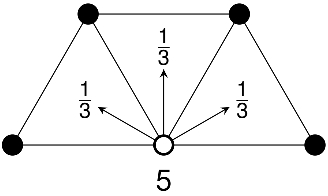
Cluster (K5a)



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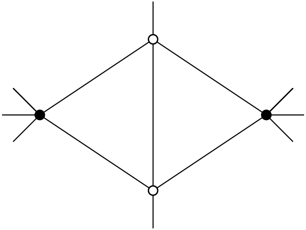

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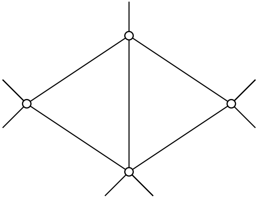

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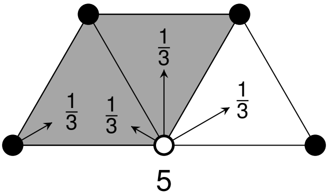

After (R1): $-\frac{4}{3}$

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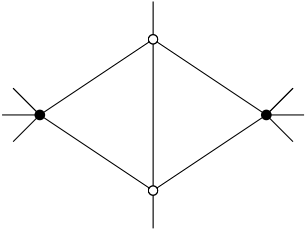

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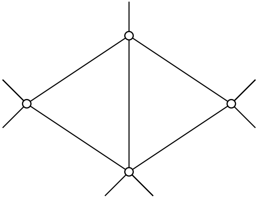

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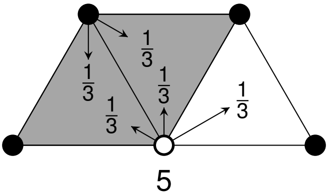
Cluster (K5a)



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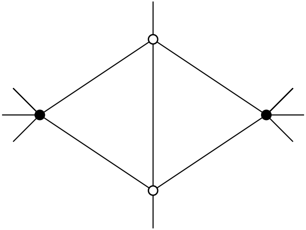

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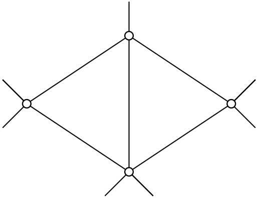

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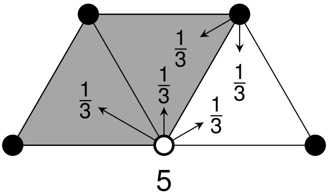
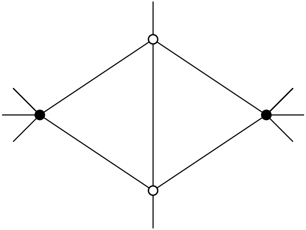
Cluster (K5a)

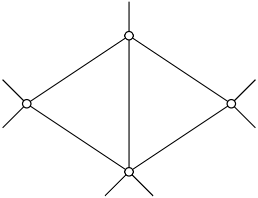


After (R1): $-\frac{4}{3}$

After (R2): $-\frac{4}{3} + 3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 0$

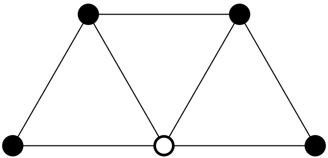


(C3)

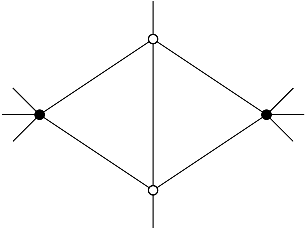


(C4)

Cluster (K5a)



After (R1): $-\frac{4}{3}$



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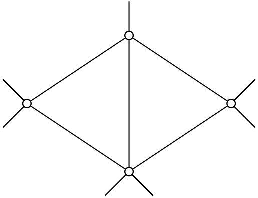
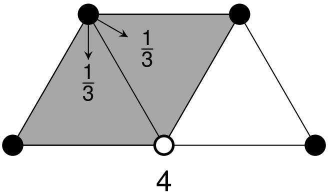
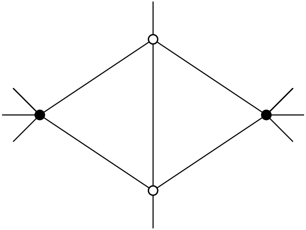


(C4)

Cluster (K5a)



After (R1): $-\frac{4}{3}$



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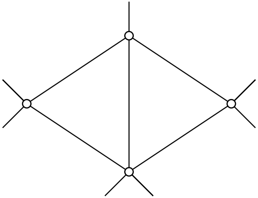
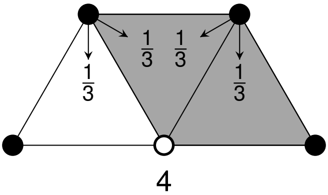


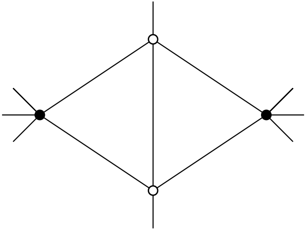
(C4)

Cluster (K5a)



After (R1): $-\frac{4}{3}$

After (R2): $-\frac{4}{3} + 4 \cdot \frac{1}{3} = 0$



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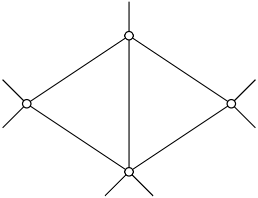

(C4)

Lessons Learned

Bad News

Fewer Faculty $=$ More Work!

Discharging needs too many details!

Know your background!

Good News

Everyone knows discharging!

Reducible configurations parallelize!

Do not settle!

$(4, 2)$-Choosability of Planar Graphs with Forbidden Structures

A Working Seminar Report

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September 8, 2015
ISU Discrete Math Seminar