A Toast to Three Russians

Derrick Stolee

October 14, 2009

Pafnuty Chebyshev

May 16, 1821 December 8, 1894

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Andrey Markov

June 14, 1856 July 20, 1922

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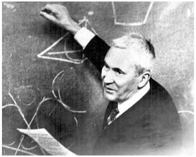


Andrey Kolmogorov

April 25, 1903 October 20, 1987

Pafnuty Chebyshev

- Father was a military officer.
- Studied at Moskow University
- Worked on probability, statistics, number theory.
- Worked with Bienaymé, Lebesgue, Cayley, Sylvester, Dirichlet...
- Died 1894.

May 16, 1821
–
December 8, 1894

Pafnuty Chebyshev

Fun Facts

- Considered to be father of Russian mathematics.
- Proved: for all n, there is a prime p with $n \leq p \leq 2n$.
- Contributed substantially to the Prime Number Theorem.

May 16, 1821
–
December 8, 1894

Chebyshev's Inequality
Measure-Theoretic Statement

Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function. Then, for any $t > 0$,

$$\mu\{x \in X : |f(x)| \geq t\} \leq \frac{1}{t^2} \int_X f^2 d\mu.$$

Chebyshev's Inequality

Probabilistic Statement

Theorem

Let X be a random variable with expected value $\mathbb{E}[X]$ and variance $\mathrm{Var}[X]$. *Then, for any* $k > 0$,

- $\Pr[|X - \mathbb{E}[X]| > k] < \frac{\mathrm{Var}[X]}{k^2}$.
- $\Pr\left[|X - \mathbb{E}[X]| > k\mathbb{E}[X]\right] < \frac{\mathrm{Var}[X]}{k^2\mathbb{E}[X]^2}$.
- *If* $X \geq 0$ *and* $\mathbb{E}[X] > 0$, $\Pr\left[X = 0\right] < \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2}$.

Chebyshev's Inequality

Combinatorial Uses

- Actually due to Irénée-Jules Bienaymé.
- Using C.I. is called *The Second Moment Method*.
- Shows a certain property holds *almost always*.

Chebyshev's Inequality
Combinatorial Example

Let $G \sim G(n,p)$. If $p \gg n^{-2/3}$, then G has a 4-clique *almost always*.
Let X be number of 4-cliques.

$$\mathbb{E}[X] = \binom{n}{4} p^6 = \omega(n^4 \cdot n^{-4}) = \omega(1) \xrightarrow{n\to\infty} \infty.$$

$$\mathrm{Var}[X] = o(n^4p^6 + n^5p^9 + n^6p^{11}) = o(n^8p^{12}) = o(\mathbb{E}[X]^2).$$

So, $\Pr[X=0] < \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \xrightarrow{n\to\infty} 0.$

Chebyshev's Inequality

Number Theory

Theorem (The Weak Law of Large Numbers)

Let $\{X_n\}_{n=1}^{\infty}$ be independent random variables with $\mathbb{E}[X_n] = \mu < \infty$ for all n. Then, the sample average $Y_N = \frac{1}{N}\sum_{n=1}^{N} X_n$ has the property

$$Y_n \xrightarrow{P} \mu.$$

i.e. If I flip a fair coin many times, it is extremely unlikely to have the number of heads be significantly different than the number of tails.

Cebyshev's Toast

If life ever hands you bad results,

just increase the sample size.

Andrey Markov

- Studied under Chebyshev
- Worked on differential equations, probability, continuous fractions.

June 14, 1856
–
July 20, 1922

Andrey Markov

Fun Facts

- Best known for Markov Chains.
- Proved "Markov Brother's Inequality" with brother Vladimir.
- Refused to be "agent of governance" during student riots.
- Requested to be Excommunicated.

June 14, 1856
–
July 20, 1922

Markov's Inequality

Measure-Theoretic Statement

Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function. Then, for any $t > 0$,

$$\mu\{x \in X : |f(x)| \geq t\} \leq \frac{1}{t} \int_X |f| d\mu.$$

Markov's Inequality

Probabilistic Statement

Theorem

Let X be a random variable with expected value $\mathbb{E}[X]$. Then, for any $k > 0$,

$$\Pr[X \geq k] < \frac{\mathbb{E}[X]}{k}.$$

Markov's Inequality
Combinatorial Uses

- Actually due to Chebyshev!
- Shows a certain property holds *almost always*.

Markov's Inequality

Combinatorial Example

Let $G \sim G(n,p)$. If $p \ll n^{-2/3}$, then G does not have a 4-clique *almost always*.

Let X be number of 4-cliques.

$$\mathbb{E}[X] = \binom{n}{4} p^6 = o(n^4 \cdot n^{-4}) = o(1) \xrightarrow{n\to\infty} 0.$$

So, $\Pr[X \geq 1] < \frac{\mathbb{E}[X]}{1} \xrightarrow{n\to\infty} 0.$

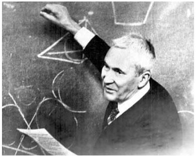
Markov's Toast

May life rarely hand you

more than you expect to handle.

Andrey Kolmogorov

- Mother died after his birth.
- Father was deported.
- Raised by his aunt.
- Worked on probability theory, topology, logic, turbulence, classical mechanics and computational complexity.



April 25, 1903
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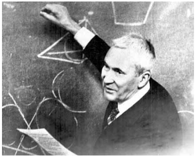
October 20, 1987

Andrey Kolmogorov
Fun Facts

- First publication was in Russian History.
- Became famous for a Fourier Series that diverges almost everywhere.
- Provided significant contributions to theory of Markov Chains.
- Frequently changed area of work entirely.



April 25, 1903
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Every mathematician believes he is ahead over all others. The reason why they don't say this in public, is because they are intelligent people

Kolmogorov Complexity

Consider the string

abcabcabcabcabcabcabcabc

How would you describe this string?

Kolmogorov Complexity

Consider the string

slghqwginvlsitalsdjtnljbviuzlidgkjgtkwasdkub

How would you describe this string?

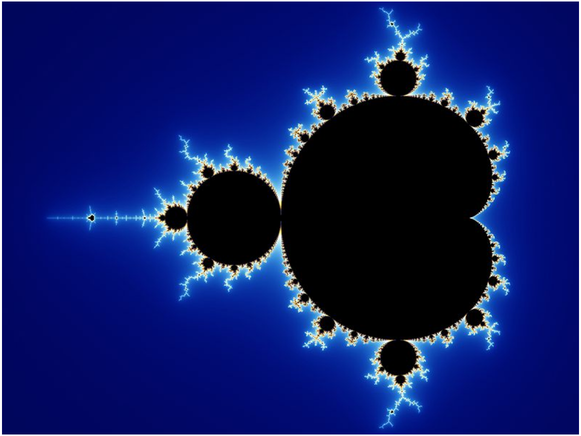
Kolmogorov Complexity

Kolmogorov Complexity takes the following ingredients:

1. An alphabet Σ
2. A descriptive language (Assembly code, Turing machine encodings)
3. A set of strings (subset of Σ^* or Σ^∞)

Then, forms a *complexity measure* K where $K(X)$ is the minimum length of a string that describes X.

Differs from Shannon information theory by focusing on computation.



$$\left\{x \in \mathbb{C} : (a_n)_{n=0}^{\infty}, a_0 = 0, a_n = a_{n-1}^2 + x, \lim_{n\to\infty} a_n = \infty\right\}$$

Kolmogorov Complexity

Big Results in the Theory

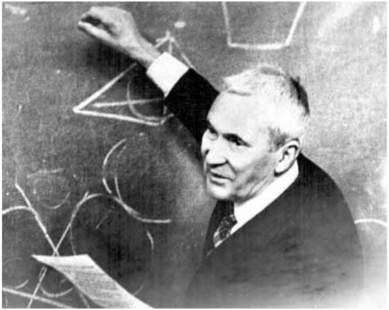
- K is incomputable.

"The smallest number that cannot be described in under twelve words."

- Chain rule:
  $K(X, Y) = K(X) + K(Y \mid X) + O(\log(K(X, Y)))$.
- Strongly related to randomness extractors (and hence, pseudorandom generators).

Kolmogorov's Toast

Live an unpredictable life



if only for the interesting biography.