Isomorph-free generation of 2-connected graphs with applications

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March 19, 2011

Computer Search

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- Generate theorems.

2-connected graphs

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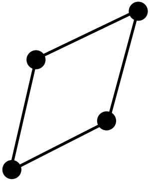
2-connected graphs are exactly the graphs with ear decompositions.

2-connected graphs and ear augmentations



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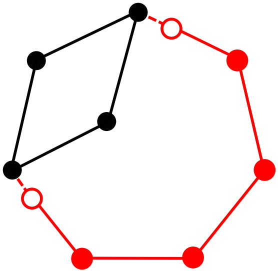

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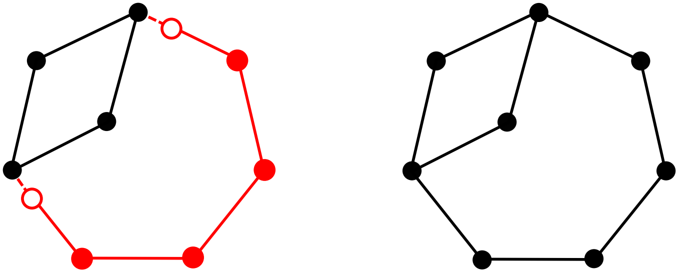

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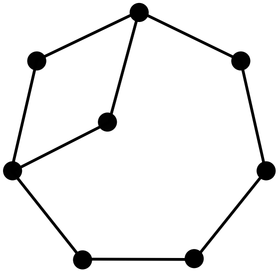

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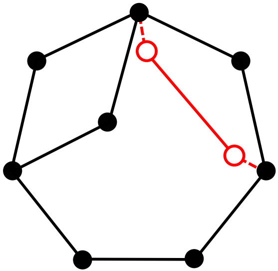

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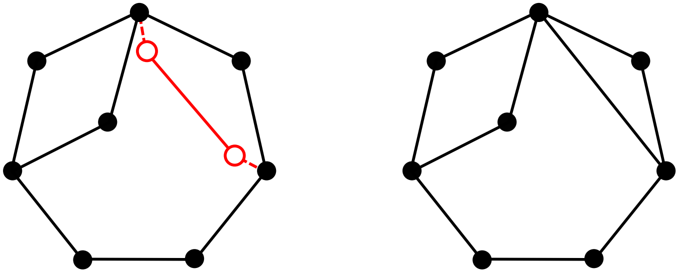

Generating by Ear Augmentations

Starting at each cycle, adding all possible ear augmentations will generate all 2-connected graphs.

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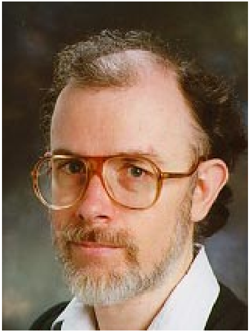
LOTS of redundancy!



Brendan McKay

"Isomorph-free Exhaustive Generation"

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Generate all graphs of a given type with each isomorphism class represented exactly once.

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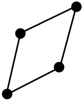
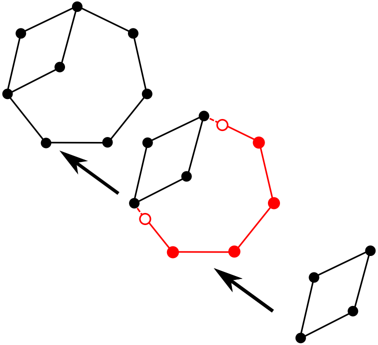
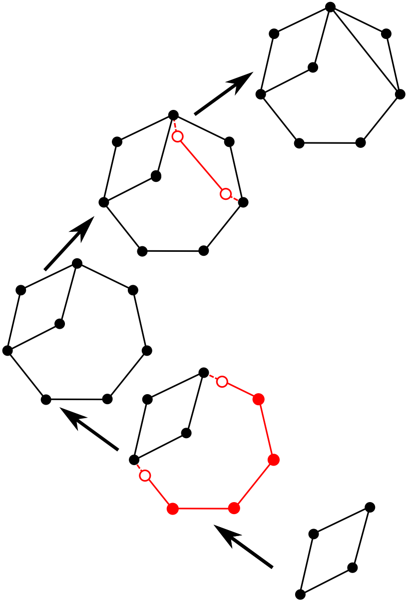
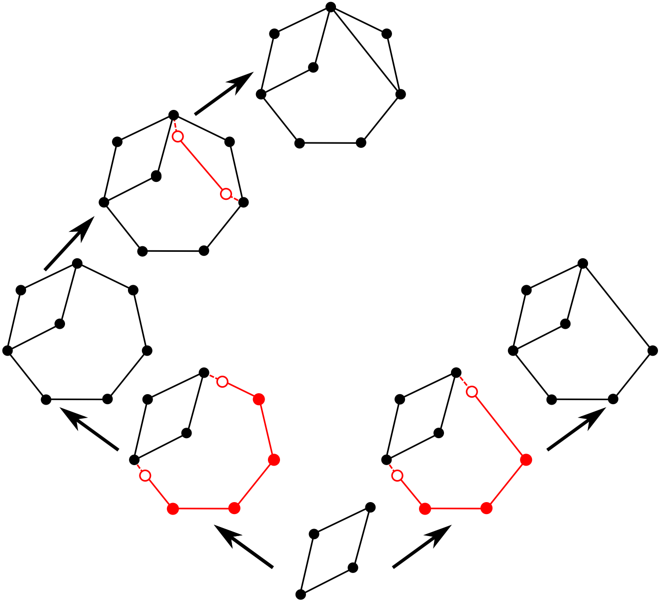
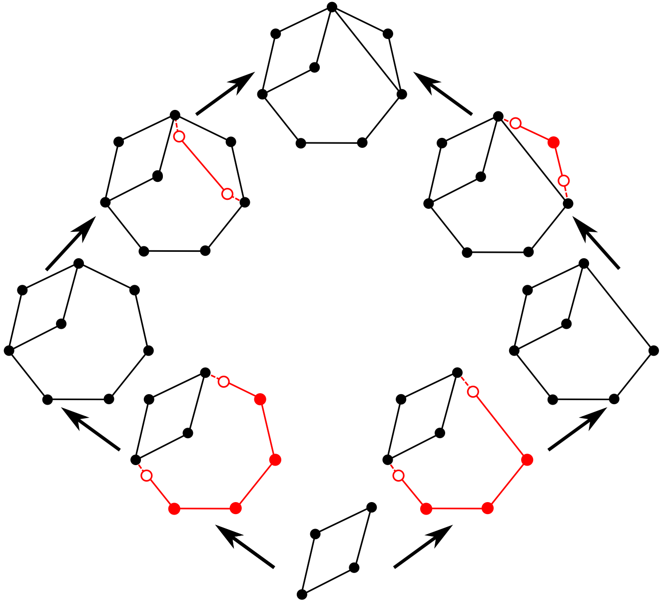
3. A pruning procedure.

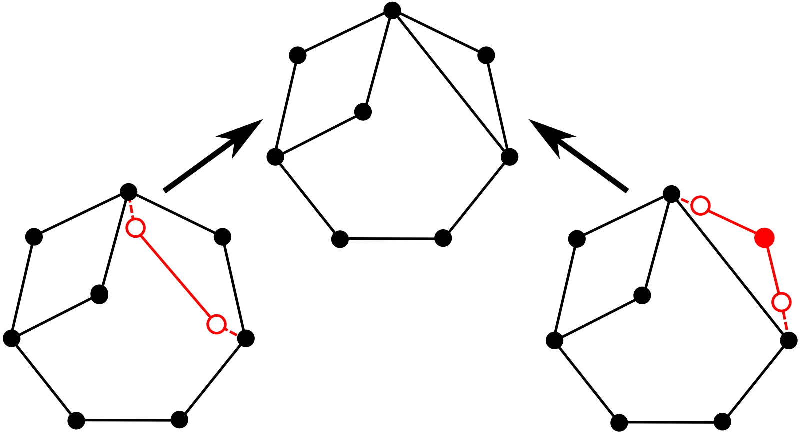


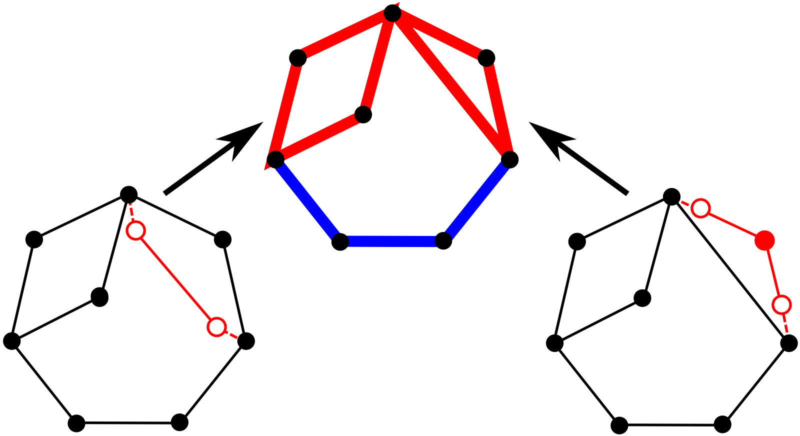


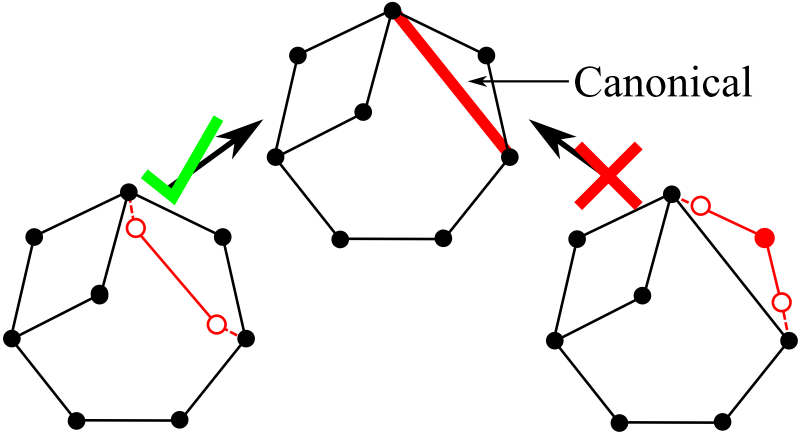


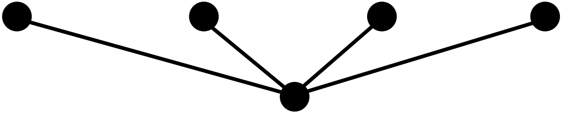


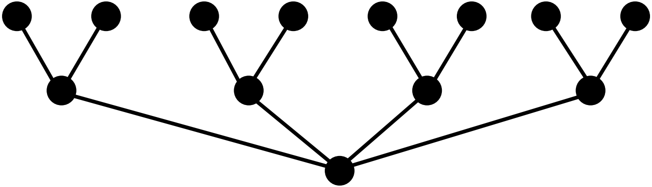


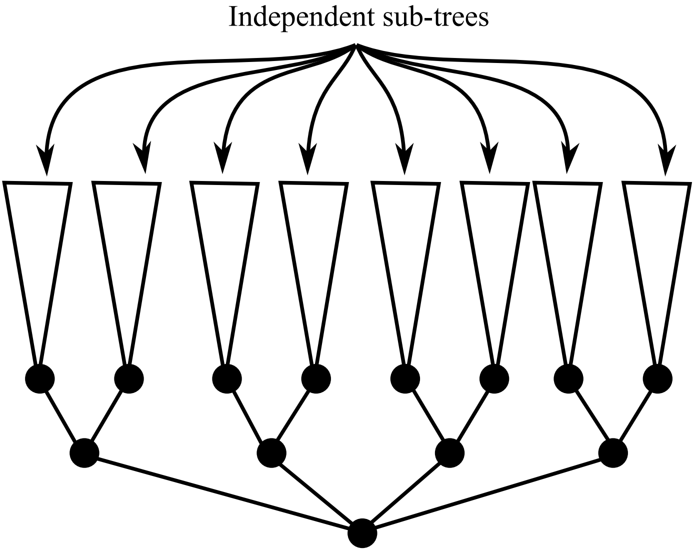





Canonical






Independent sub-trees

Implementation

Implemented in the TreeSearch library for parallelization in the Condor scheduler.

Executed on the Open Science Grid, a collection of supercomputers around the country.

Open Science Grid

Generating 2-connected Graphs

N	C_N	CPU time
5	10	0.01s
6	56	0.11s
7	468	0.26s
8	7123	10.15s
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Slower than vertex-augmentations, faster than edge-augmentations.

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3. p-Extremal Graphs
    1. Sparse family.
    2. Ear-monotone constraints.
    3. Structure of search.

Application 3: p-Extremal Graphs

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Let $\Phi(G)$ denote the number of perfect matchings in a graph G.

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Definition

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Question

How does $f(n, p)$ behave, and which graphs are p-extremal?

 

Andrzej Dudek John Schmitt

“On the Size and Structure of Graphs with a Constant Number of 1-Factors”

Dudek & Schmitt

Dudek & Schmitt

1. If G has p perfect matchings, n_0 vertices and $\frac{n_0^2}{4} + c$ edges, then for all $n \geq n_0$, $f(n,p) \geq \frac{n^2}{4} + c$.

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3. Computed c_p for $p \in \{1, \ldots, 6\}$.
4. Found structure for p-extremal graphs with $p \in \{1, 2, 3\}$.

Stephen G. Hartke

Derrick Stolee

Douglas B. West

Matthew Yancey

“On extremal graphs with a given number of perfect matchings”

Hartke, Stolee, West, & Yancey

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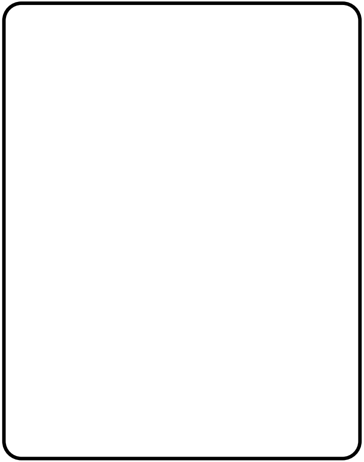
p	1	2	3	4	5	6	7	8	9	10
c_p	0	1	2	2	2	3	3	3	4	4
n_p	2	4	4	6	6	6	6	6	6	6
	[DS10]						[HSWY11+]			

The Structure of p-Extremal Graphs

p-Extremal Graph



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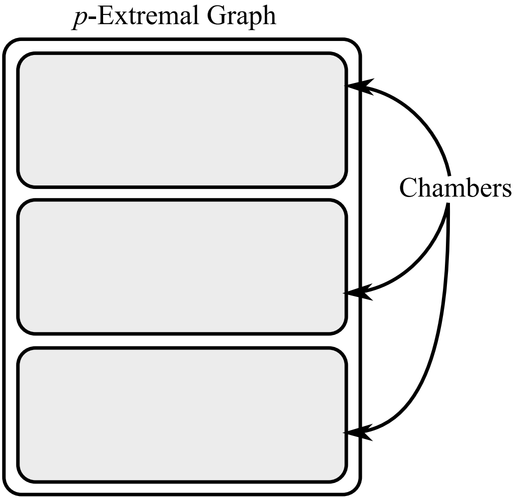

The Structure of p-Extremal Graphs

Chambers are the connected components in the subgraph of edges appearing in perfect matchings (*allowable* edges).

For G a graph with chambers $G_1, \dots, G_k$,

$$\Phi(G) = \prod_{i=1}^{k} \Phi(G_i).$$

$$c(G) \leq \sum_{i=1}^{k} c(G_i).$$

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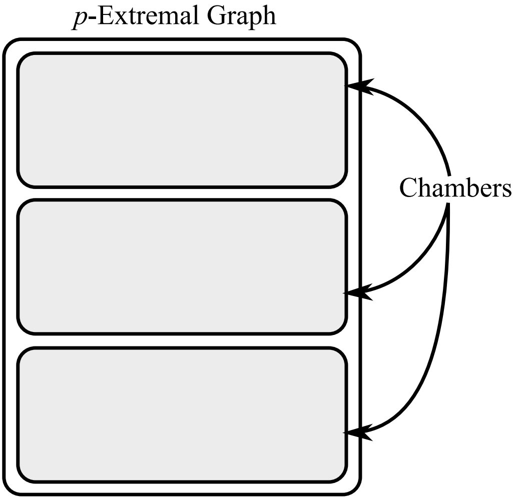

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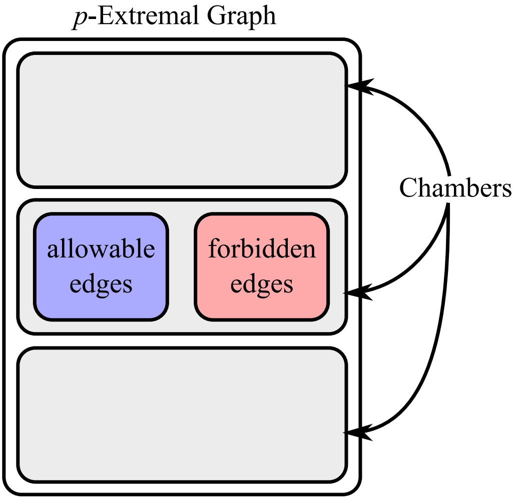

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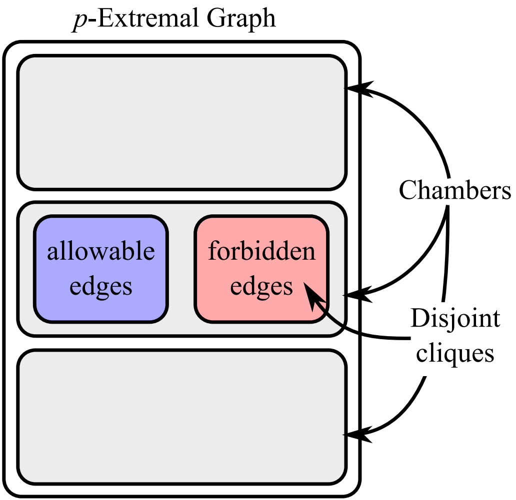

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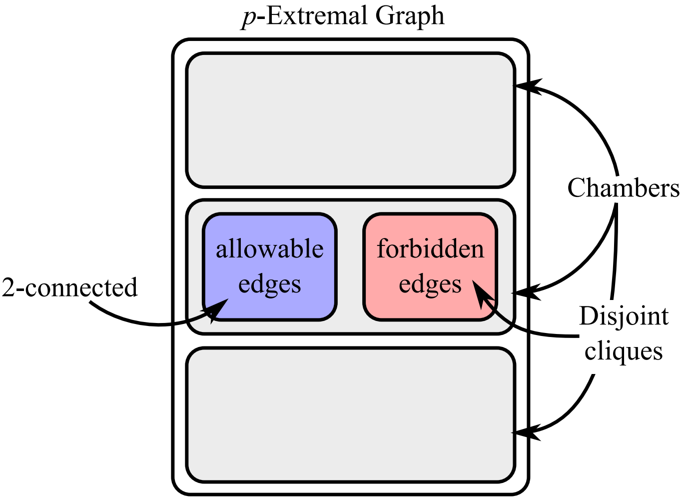

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If H is a 1-extendable graph, there is a graded ear decomposition $H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_k$ *So that*

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Graphs which appear "between" two 1-extendable graphs in a two-ear augmentation are *almost 1-extendable* graphs.

The Search Space

Input: p, N, c.

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Solutions: Chambers G with p perfect matchings and $c(G) \geq c$.

Canonical Deletion

Find an ear in H to delete by priority:

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Almost 1-extendable

↓

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Almost 1-extendable → 1-extendable	1-extendable → 1-extendable
Unique such choice	Minimum length ear
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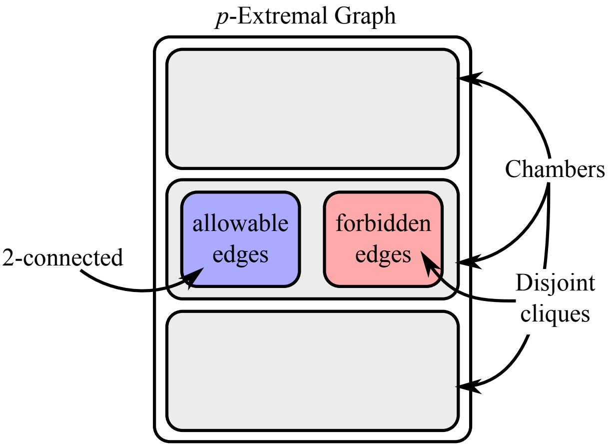
Almost 1-extendable	1-extendable	1-extendable
↓	↓	↓
1-extendable	1-extendable	Almost 1-extendable
Unique such choice	Minimum length ear	Minimum length ear
	Minimum canonical label of endpoints	Minimum canonical label of endpoints

Finding Solutions

Question

How do we transition from 1-extendable graphs to extremal chambers?

Finding Solutions



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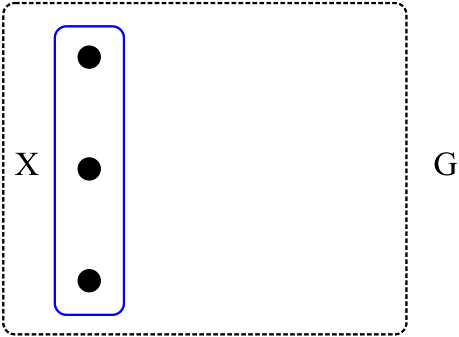

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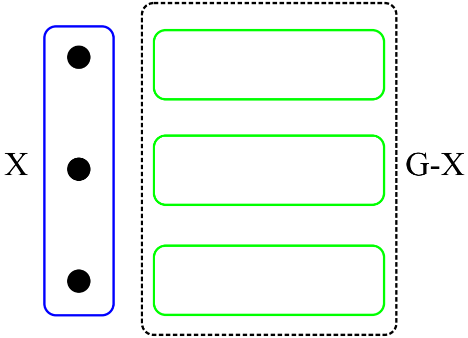

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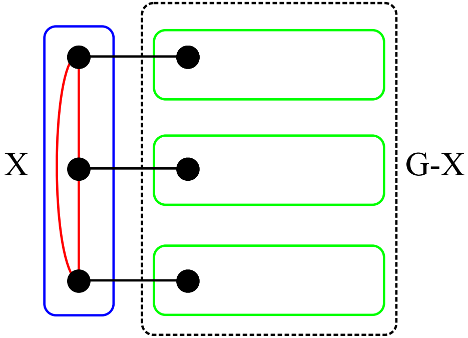

Finding Solutions

Definition

A *barrier* in a graph G with $\Phi(G) > 0$ is a set $X \subset V(G)$ so that $c_o(G - X) = |X|$.

In a p-extremal chamber, every barrier is a clique of forbidden edges.

Conflicting Barriers

Two barriers X, Y *conflict* if:

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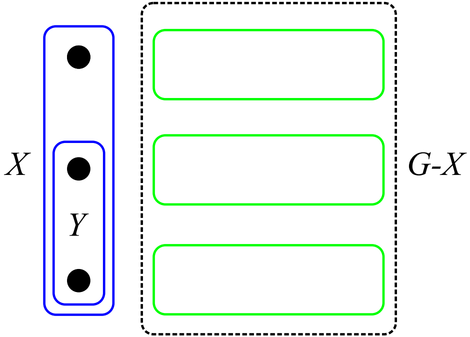

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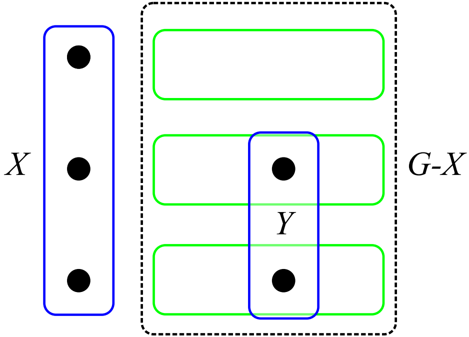

Finding Solutions

$$\{\text{Maximal chamber supergraphs of } H\}$$

$$\Longleftrightarrow$$

$$\{\text{Maximal sets of non-conflicting barriers in } H\}$$

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We want to prune when the excess can never reach c, no matter what augmentations we use.

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$$
\begin{array}{lccc}
\textbf{1-extendable:} & H & \xrightarrow{\text{ear-augmentations}} & H' \\
& \text{best filling} \downarrow & & \downarrow \text{best filling} \\
\textbf{Chambers:} & G & & G'
\end{array}
$$

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$$\begin{aligned} c(G') \leq c(G) & \\ & + 2(\Phi(H') - \Phi(H)) \\ & - \frac{1}{4}(n(H') - n(H))(n(H) - 2). \end{aligned}$$

Pruning and Optimizations

Let H be the current 1-extendable graph and G a best filling.

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   We do not need to augment beyond N vertices.
3. If adding an ear at endpoints x, y increases $\Phi(H)$ beyond p, never augment on that pair again.

Results

p	1	2	3	4	5	6	7	8	9	10
c_p	**0**	**1**	**2**	**2**	**2**	**3**	**3**	**3**	**4**	**4**
n_p	2	4	4	6	6	6	6	6	6	6

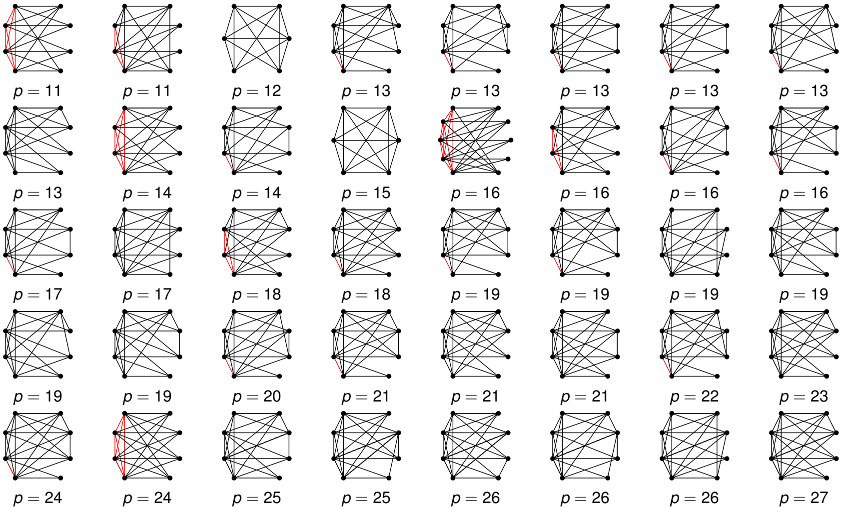
p	11	12	13	14	15	16	17	18	19	20
c_p	3	**5**	3	4	**6**	4	4	5	4	5
n_p	8	6	8	8	6	8	8	8	8	8

p	21	22	23	24	25	26	27			
c_p	5	5	5	**6**	5	5	**6**			
n_p	8	8	8	8	8	8	8			

Timing

p	N_p	c_p	Total CPU Time				
11	14	3					43.29s
12	14	5					44.01s
13	14	3				6m	39.80s
14	16	4				12m	10.40s
15	16	6				12m	42.72s
16	16	4			2h	07m	58.60s
17	16	4			6h	46m	07.72s
18	18	5			11h	45m	01.95s
19	18	4		2d	17h	12m	31.85s
20	18	5		4d	05h	28m	11.79s
21	18	5		13d	17h	29m	12.45s
22	20	5		42d	20h	40m	30.41s
23	20	5		118d	07h	38m	36.84s
24	20	6		209d	10h	09m	54.98s
25	20	5	2y	187d	21h	48m	46.31s
26	20	5	7y	75d	13h	55m	10.27s
27	22	6	10y	247d	21h	03m	13.94s

p-Extremal Chambers



Future Work

For p-extremal graphs:

1. Find a “strong” upper bound on c_p for an infinite family of values of p.
2. A start: prove the complete graphs on $2t$ vertices are p-extremal for $p = (2t-1)!!$.
3. A lower bound on c_p which grows in the limit.

Future Work

For the technique:

1. More applications!
2. More optimizations?

Future Work

For the tools:

1. Implement vertex/edge/leaf augmentations.
2. Apply to new problems.
3. Compare to current applications.

Isomorph-free generation of 2-connected graphs with applications

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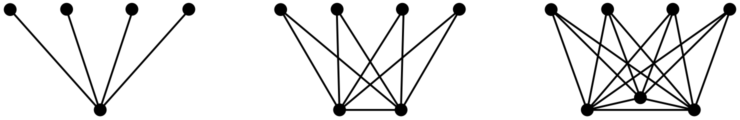
March 19, 2011

[1] Supported by NSF grants CCF-0916525 and DMS-0914815.

Application 1: Uniquely K_r-Saturated Graphs

Definition

A graph G is *uniquely K_r-saturated* if G contains no K_r and for every edge $e \in \overline{G}$ admits exactly one copy of K_r in $G + e$.



(a) 1-book (b) 2-book (c) 3-book

Figure: The $(r-2)$-books are uniquely K_r saturated.

Joshua Cooper

Paul Wenger

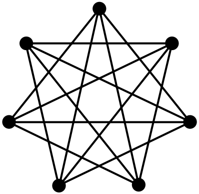
Two Conjectures:

1. For each r, there are a finite number of uniquely K_r-saturated graphs with no dominating vertex.
2. For each r, every uniquely K_r-saturated graph with no dominating vertex is regular.

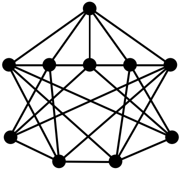
Previously verified to 9 vertices.

Application 1: Uniquely K_r-Saturated Graphs

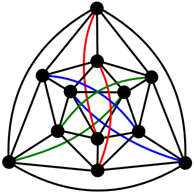
1. Uniquely K_r-saturated graphs have diameter 2 (and are 2-connected).
2. Strength: K_4-free is a sparse, monotone property.
3. Verified for $r = 4$ and $n \leq 12$.
4. Verified for $r \in \{5, 6\}$ and $n \leq 11$.



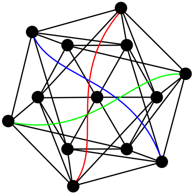
(a)



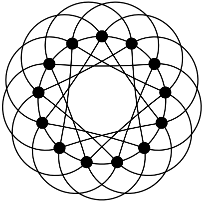
(b)



(c)



(d)



(e)

New! Joint with Stephen G. Hartke