MATH 482, Spring 2013 - Homework 4

Assigned Monday 10/07. Due Wednesday 10/09.

Assigned: 1.e, 2.d, 3.b, 4.a.

1. Find a maximum-weight perfect matching and a minimum-weight vertex cover for the bipartite graphs with weight matrices given below. (*Assigned: e*) [In blue]

$$\begin{array}{cc} & \begin{matrix}2&4&2&0\end{matrix} \\ \begin{matrix}+3&4\\+6&7\\0&3\\3&7\end{matrix} & \begin{bmatrix}6&\boxed{8}&6&3\\3&9&9&\boxed{7}\\\boxed{5}&0&4&1\\0&3&\boxed{9}&7\end{bmatrix} \\ & \begin{matrix}-3&0&3&0\end{matrix}\end{array}$$

a.

$$\begin{array}{cc} & \begin{matrix}4&2&0&3&4\end{matrix} \\ \begin{matrix}1&8\\3&4\\1&4\\1&2\\5&6\end{matrix} & \begin{bmatrix}3&1&\boxed{8}&5&9\\\boxed{8}&5&1&5&8\\5&\boxed{6}&2&7&1\\3&4&2&3&\boxed{6}\\8&\cancel{8}&3&\boxed{9}&8\end{bmatrix} \\ & \begin{matrix}2&0&2&2&0\end{matrix}\end{array}$$

b.

$$\begin{array}{cc} & \begin{matrix}2&4&0&5&0\end{matrix} \\ \begin{matrix}0&3\\4&5\\1&9\\1&3\\1&7\end{matrix} & \begin{bmatrix}5&7&3&\boxed{8}&0\\4&9&\boxed{5}&8&4\\1&8&9&1&\boxed{9}\\1&\boxed{7}&2&8&2\\\boxed{9}&6&1&1&6\end{bmatrix} \\ & \begin{matrix}0&5&0&0&0\end{matrix}\end{array}$$

c.

$$\begin{array}{cc} & \begin{matrix}1&3&0&1&4&4\end{matrix} \\ \begin{matrix}4&5\\1&6\\0&9\\4&8\\1&5\\4&8\end{matrix} & \begin{bmatrix}6&4&4&4&\boxed{9}&9\\5&\boxed{9}&2&1&4&7\\1&0&\boxed{9}&6&3&9\\\boxed{9}&5&4&6&7&9\\5&8&5&1&4&\boxed{9}\\5&5&8&\boxed{9}&7&8\end{bmatrix} \\ & \begin{matrix}1&0&0&0&3&4\end{matrix}\end{array}$$

d.

$$\begin{array}{cc} \begin{matrix}1&6\\0&9\\0&4\\1&5\\0&1\\0&9\end{matrix} & \begin{bmatrix}8&1&3&\boxed{9}&4&3\\3&\boxed{9}&0&8&8&4\\\boxed{9}&1&4&1&0&0\\8&3&\boxed{5}&8&6&1\\6&1&0&0&\boxed{5}&0\\2&1&0&7&0&\boxed{9}\end{bmatrix} \\ & \begin{matrix}5&0&0&3&4&0\\2&0&0&0&0&0\end{matrix}\end{array}$$

e.

$$\begin{array}{cc} \begin{matrix}1&3\\2&6\\1&7\\0&6\\0&5\\6&6\end{matrix} & \begin{bmatrix}6&3&4&\boxed{4}&1&6\\7&2&8&7&5&\boxed{9}\\2&\boxed{7}&3&3&2&1\\\boxed{9}&0&1&0&7&8\\7&0&0&6&\boxed{6}&5\\9&6&\boxed{9}&6&7&6\end{bmatrix} \\ & \begin{matrix}3&0&3&1&1&3\\1&0&0&0&0&0\end{matrix}\end{array}$$

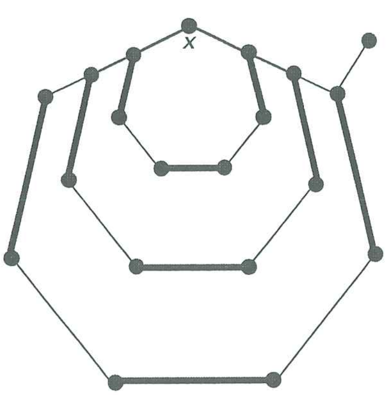
f.

Values

a.	29	12
b.	37	13
c.	38	12
d.	54	22
e.	46	4
f.	44	11

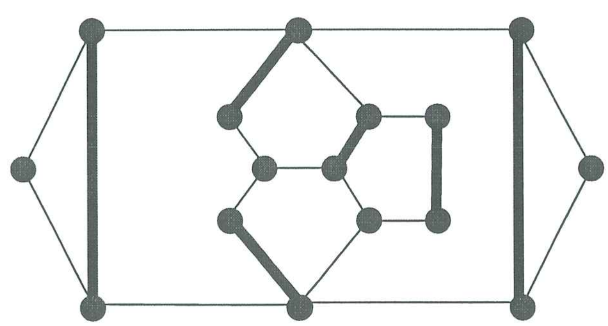
2. Find a minimum-weight perfect matching and a maximum-weight vertex under-cover for the bipartite graphs with weight matrices given above. (*Assigned: d*) (in Red)

3. In the graphs below, perform the blossom algorithm with the given matchings to either find a perfect matching or a set A such that $\mathrm{oc}(G - A) > |A|$.



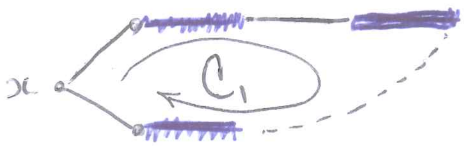
a.

(Use a tree rooted at x for this problem.)

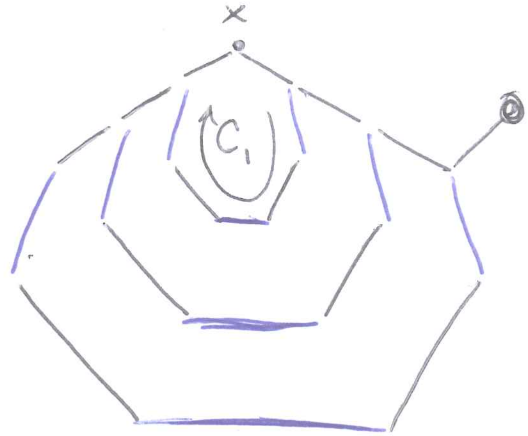


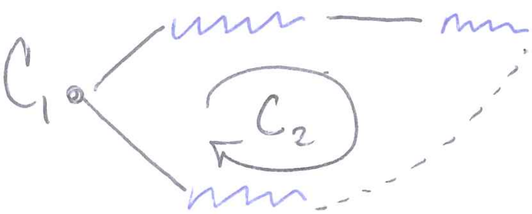
b. (*Assigned!*)

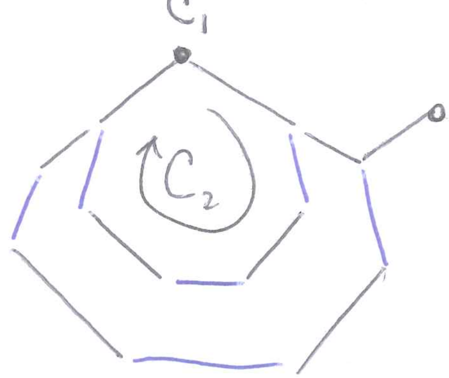
3.a. Preferred Method Below (There also exists an M-augmenting path, which can be discovered by carefully building the M-alternating tree.)

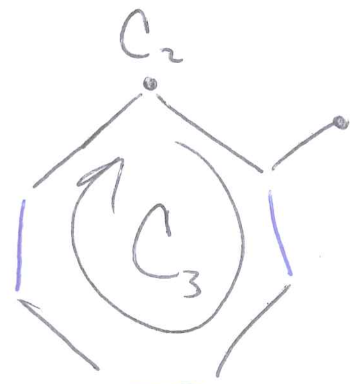


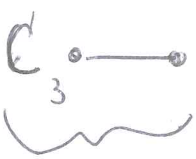
Stable, not frustrated!



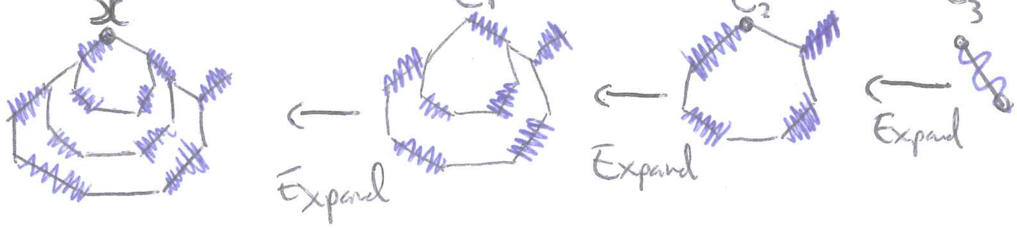




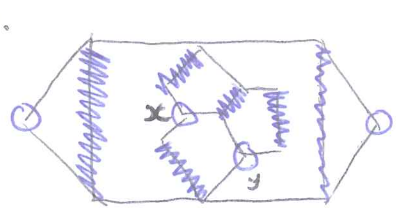




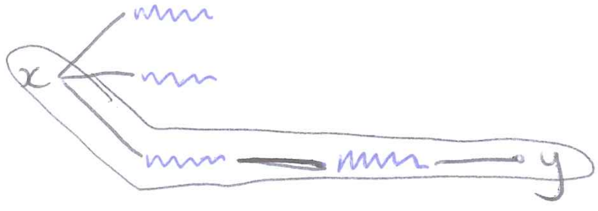
M-augmenting!



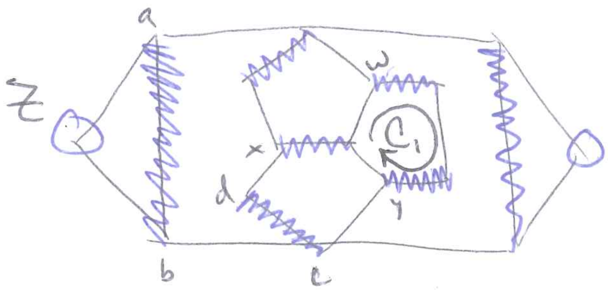
3.b.



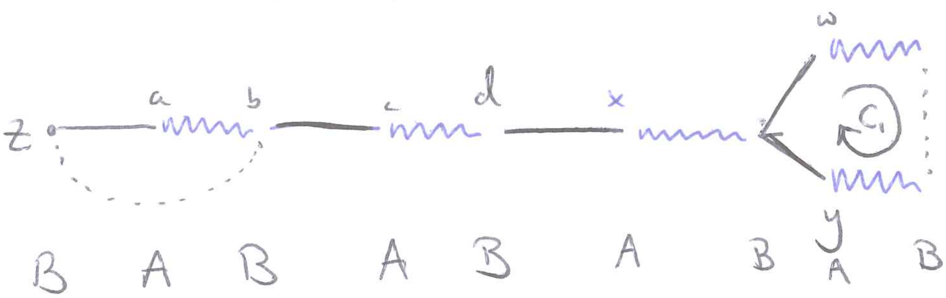
There are 4 unsaturated vertices, so the first step is to find a better matching (which is possible in this case.



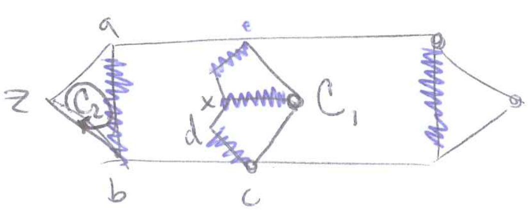
→ M-augmenting!

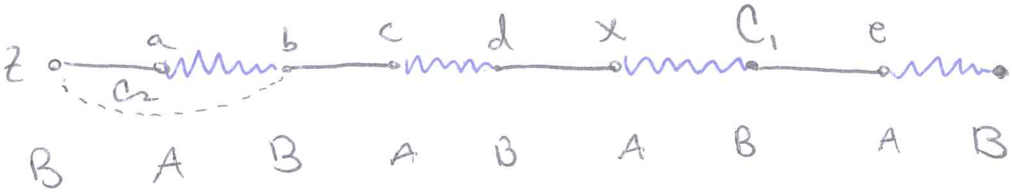


Still not perfect, build an M-alternating tree and try to grow the matching!

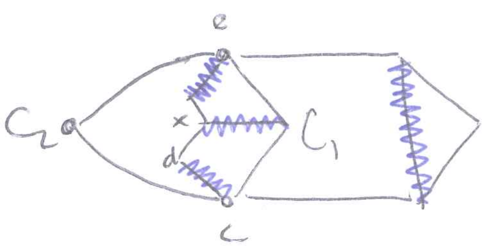


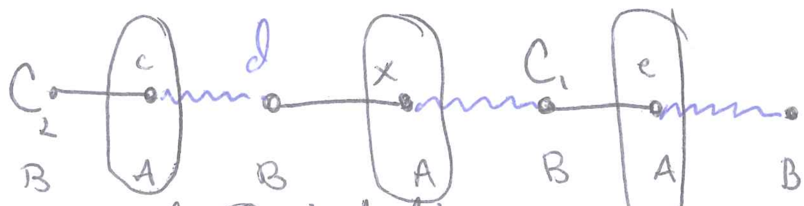
Stable, not frustrated.
Shrink on C_1





Stable, not frustrated
Shrink on C_2



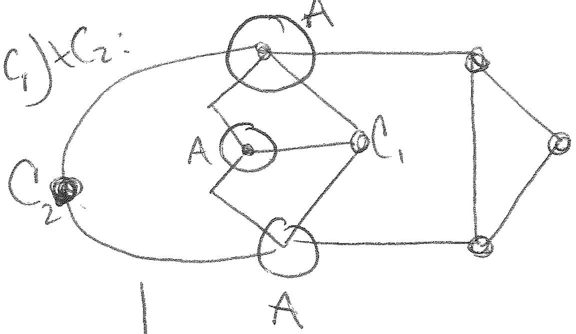


Stable and Frustrated!

3.b. (Continued.)

In Shrunken Graph:

$(G \times C_1) \times C_2$:

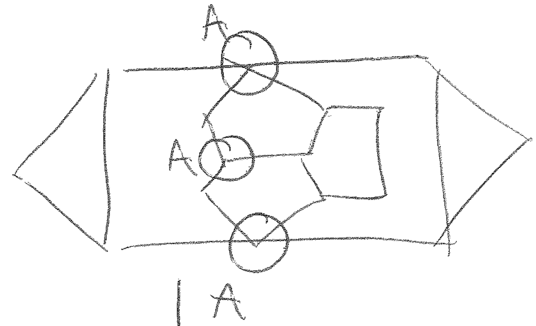


$((G \times C_1) \times C_2) - A$:

~~Four~~ odd components!

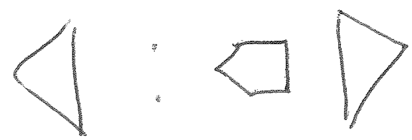
In Full Graph:

G:



$\downarrow A$

$G - A$



Five odd components, but $|A| = 3$.

$o(G-A) > A$

So no perfect matching!

4. Let X and Y be sets of size n where every element $x \in X$ has a total ranking of the elements of Y, and every $y \in Y$ has a total ranking of the elements of X. Let M be a stable matching found by the Gale-Shapely proposal algorithm with the elements of X proposing, and let M' be any stable matching.

a. (*Assigned!*) Prove that for every $x \in X$, x prefers its match in M to its match in M'.

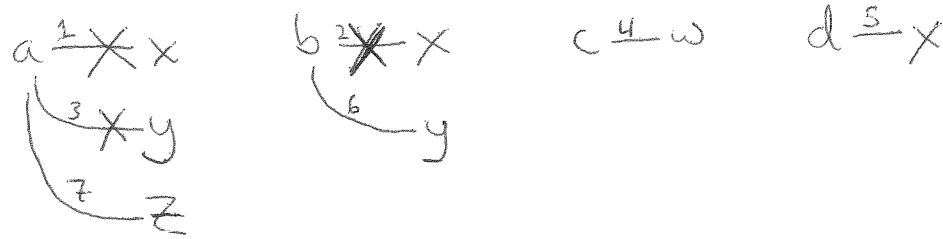
Proof. Let M be any stable matching and let xy be a matched pair in M. We will show that as we run the Gale-Shapely Algorithm that y will never reject x. (Thus, if x proposes to y, they will remain matched. Otherwise, x will end up matched to some y' that is strictly preferrable.)
Run the Gale-Shapely algorithm until the first rejection of this type occurs: some y rejects the proposal from x where $xy \in M$. This implies that y prefers some x' to x. Let y' be the vertex such that $x'y' \in M$. Since y' did not reject x' (remember: this is the first rejection of this type) it must be that x' has not yet proposed to y'. But this implies that x' prefers y to y' and since y prefers x' to x, we have that $x'y$ is an unstable pair for M and M is unstable, a contradiction! $\square$

(*Observe:* Many people would think to look at the matching that results and create a list of vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ such that the proposal algorithm matching M has $x_i y_i \in M$ and there is some other matching M' where $x_i y_{i+1} \in M'$ and x_i prefers y_{i+1} to y_i. Unfortunately, this proof strategy is extremely unlikely to work, and likely has a big, subtle flaw.)

b. Prove that for every $y \in Y$, y prefers its match in M' to its match in M.

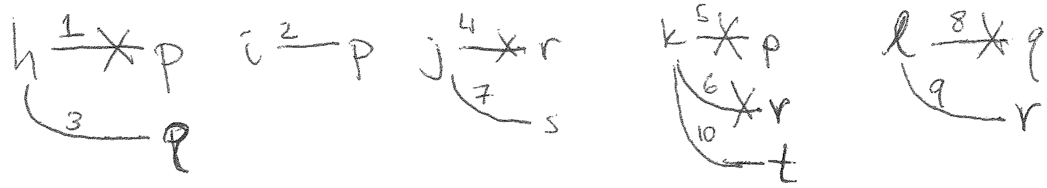
Proof. Let M be any stable matching and let xy be a matched pair in M. By the proof above, observe that as we run the Gale-Shapely Algorithm that y will never reject x. This implies that y is never proposed to by any of its preferences to x, and so either y is matched to x or to an x' where y strictly prefers x over x'. $\square$

5. a. (Proposals numbered by order.)



Stable: az, by, cw, dx.

5. b.



Stable: hq, ip, js, kt, lr.