MATH 566
Discrete Optimization

Notes by Derrick Stolee
dstolee@iastate.edu

Book: Combinatorial Optimization

William J. Cook
William H. Cunningham
William R. Pulleyblank
Alexander Schrijver

Course Plan:

Introduction

- Combinatorial Problems
- Tools
- Approach (Certificates, Proof of Optimality)

Linear Programming (Appendix A)

- Forms and Equivalence
- Basic Feasible Solutions
- Duality (S.O.B. Method)
- Solving by Computer ☆

Maximum Flows on Digraphs

- Formulation
- Max Flow / Min Cut
- Ford-Fulkerson Algorithm ☆
- Undirected Graphs + Applications

Maximum Matchings

- Formulation
- Duality: Vertex Covers
- Hungarian Algorithm for Bipartite Graphs ☆
- Hall's Theorem

* Edmond's Blossom Algorithm

Integer Programming

- Formulation & Linear Relaxation
- Total Unimodularity
- Total Dual Integrality
- Cutting Planes ☆
- Branch-and-Bound ☆

Traveling Salesman Problem

- Formulation
- Heuristics + Local Search *
- Branch + Bound *

Matroids

- Definition and Greedy Algorithm
- Matroid Intersection Theorem
- Applications
- Weighted Matroids

* Combinatorial Generation

- Isomorphisms and Automorphisms
- Orbital Branching
- Canonical Deletion
- Feasibility Pump

[redacted]

Introduction:

Discrete/Combinatorial:

Finite objects.

Graphs, Set Systems,

(Q: What do they model?) Networks, circuits, road maps, business processes, data-flow diagrams (code) Chemicals

Subobjects:

(Q: What Subobjects) Paths, Trees, Flows, Cuts (Edge+Vertex), Matchings ✓ Separators, Decompositions, Indep. Sets.

Extremal vs. Optimal

Extremal: Best object among a class of objects

Optimal: Best subobject within a given object.

Input: Object
Output: Subobject + Value.

Certificates: Proof by Example.

"Here is the (short) evidence!"

Two parts: 1. Here is why this is good.

2. Here is why we can do no better! (Duality)

Generality and Inter-connectedness

Solving one problem may provide solution to another.

Most problems will be expressable as a Linear Program (LP) or integer program (IP)

A is an $\underbrace{m}_{\text{rows}} \times \underbrace{n}_{\text{columns}}$ matrix, $\underline{b} \in \mathbb{R}^m$, $\underline{c} \in \mathbb{R}^n$, and

$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ an n-vector of variables.

Standard Form	Canonical Form	General Form
$\min \ \underline{c}^T \underline{x}$ s.t. $A\underline{x} = \underline{b}$ $\underline{x} \geq \underline{0}$	$\min \ \underline{c}^T \underline{x}$ s.t. $A\underline{x} \geq \underline{b}$ $\underline{x} \geq \underline{0}$	$\min/\max \ \underline{c}^T \underline{x}$ s.t. $A\underline{x} = \underline{b}$ $A'\underline{x} \geq \underline{b}'$ $A''\underline{x} \leq \underline{b}''$ $x_i \geq 0 \quad i \in I$ x_i free $\quad i \in I'$ $x_i \leq 0 \quad i \in I''$
Slack variables $[A \mid I]\begin{pmatrix} x \\ s \end{pmatrix} = b$	$\begin{bmatrix} A \\ -A \end{bmatrix} x \geq \begin{pmatrix} b \\ -b \end{pmatrix}$ Negation & Double inequalities	$x_i^+ - x_i^-$

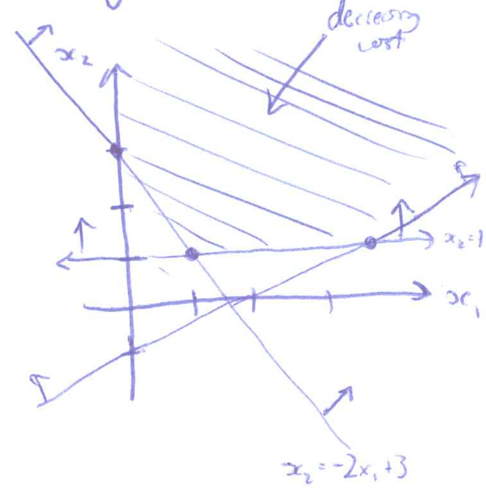
~~Ex:~~

Slack variables measure how much slack is given in a constraint.

Ex's:

$$\min \ x_1 + 2x_2 - x_3$$
$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 10 \\ 2x_2 - x_4 &= 5 \\ x_3 + 3x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$$\min \ x_1 + x_2$$
$$\text{s.t.} \quad \begin{aligned} 2x_1 + x_2 &\geq 3 \\ x_2 &\geq 1 \\ -x_1 + 2x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$



HW: Write the following LP in canonical & standard form.

$$
\begin{array}{llr}
\max & 3x_1 - 2x_2 + x_4 & \\
\text{s.t.} & x_1 + x_2 - x_3 & \geq 1 \\
 & 2x_2 + x_4 & = 0 \\
 & x_3 - 3x_4 & \leq 6 \\
 & x_1,\ x_2 & \geq 0 \\
 & x_3 & \leq 0 \\
 & x_4 & \text{free}
\end{array}
$$

~~About Solving LPs:~~

~~Use Standard form. Allows knowledge of solving systems of linear equality.~~ ~~A is full rank $\to m \leq n$~~

~~Define: feasible point. (Convexity) (Extremal pts)~~

~~basic feasible point (bfs).~~

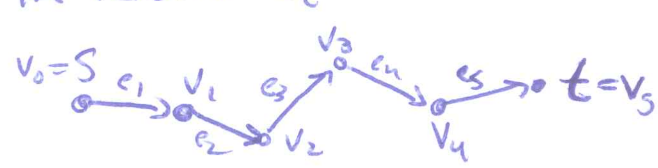
Shortest Paths as an LP

For a (directed) graph G, weight function $w(ij) \geq 0$ $ij \in E(G)$ and $s, t \in V(G)$, an

Vertex-centric:
- st-path is a list $v_0, \dots, v_k$ of vertices s/t
- $v_0 = s$, $v_k = t$, and $v_{i-1}v_i \in E(G)$ for all $i \in \{1, \dots, k\}$.

tail $\longrightarrow$ head

Edge-centric:
- An st-path is a list $e_1 \dots e_k$ of edges s/t
- s is the tail of e_1
- t is the head of e_k
- the head of e_i is the tail of e_{i+1} for all $i \in \{1, \dots, k-1\}$.



$wt(P) = \sum_{e \in P} w(e)$.

Fix $V(G) = \{1, \dots, n\}$, $s=1$, $t=n$

For each edge $ij \in E(G)$, let $x_{i,j} \geq 0$ be a real variable, corresponding to "how much of" ij to use in the path".

$x_{i,j} = 0 \Rightarrow$ none of the edge.
$x_{i,j} = 1 \Rightarrow$ all of the edge.
} Restricting to these options makes it an IP (Hard)

$x_{i,j} > 1$ will not make sense

$x_{i,j} \in (0,1)$ may make sense.

Want to minimize distance/weight, so

$$\min \sum_{ij \in E(G)} w(ij)$$

Want "one" edge leaving s,

$$\sum_{j : sj \in E(G)} x_{s,j} = 1$$

Want "one" edge ~~leaving~~ entering t,

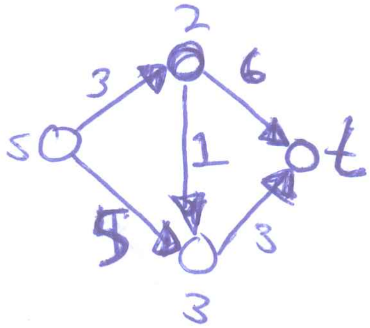
$$\sum_{i : it \in E(G)} x_{i,t} = 1$$

Want other vertices to have same # of entering and leaving edges:

$$\forall k \in \{2, \dots, n-1\}: \quad -\sum_{j : kj \in E} x_{k,j} + \sum_{i : ik} x_{i,k} = 0$$

All variables nonnegative:

$$x_{i,j} \geq 0 \quad \forall ij \in E(G)$$



$E = \{ s2, s3, 23, 2t, 3t \}$

Standard form!

$$
\begin{aligned}
\min \quad & 3x_{s,2} + 5x_{s,3} + 1x_{2,3} + 6x_{2,t} + 3x_{3,t} & & \\
\text{s.t.} \quad & x_{s,2} + x_{s,3} = 1 & & (s) \\
& x_{s,2} - x_{2,3} - x_{2,t} = 0 & & (2) \\
& x_{s,3} + x_{2,3} - x_{3,t} = 0 & & (3) \\
& x_{2,t} + x_{3,t} = 1 & & (t) \\
& x_{s,2},\ x_{s,3},\ x_{2,3},\ x_{2,t},\ x_{3,t} \geq 0 & &
\end{aligned}
$$

Optimal: $\underline{x} = (1, 0, 1, 0, 1)$

$P = \underline{s, 2, 3, t}$.

Cycles

HW

Q: • Can a feasible solution ever have $x_{i,j} > 1$?

Q: • Can an optimal solution ever have $x_{i,j} > 1$?

Q: • Is every optimal solution always 0/1-valued?

Q: • Is some optimal solution 0/1-valued?

Convex Combinations

$\underline{v}^{(1)}, \ldots, \underline{v}^{(k)} \in \mathbb{R}^n$ $\quad \sum_{i=1}^{k} \lambda_i \underline{v}^{(i)}$

$\lambda_1, \ldots, \lambda_k \in \mathbb{R}, \quad \sum \lambda_i = 1$

Combined!

A bit about solving LP's.

Three Outcomes: (Ask)

1. Exists an optimal solution $\underline{x}^*$.
2. Infeasible
3. Unbounded.

Now, consider an LP in Standard Form:

$$\min \ \underline{c}^T \underline{x}$$

s.t.

$$A\underline{x} = \underline{b}$$

$$\underline{x} \geq \underline{0}$$

Assumption: A has full rank (i.e. these m rows are linearly independent.)

(If not, then some constraints are redundant or contradictory.)

If A has full rank, then there exist sets of m columns of A that are linearly indep.

Let $\mathcal{B} = \{A_{j_1}, \dots, A_{j_m}\}$ be a basis of cols from A.

For the basis $\mathcal{B}$, the matrix $B = [A_{j_1} A_{j_2} \cdots A_{j_m}]$ is $m \times m$, invertible.

So there exists a unique solution to

$$\boxed{B \underbrace{\underline{x}_{\mathcal{B}}}_{\begin{pmatrix} x_{j_1} \\ x_{j_2} \\ \vdots \\ x_{j_m} \end{pmatrix}} = \underline{b}}$$

Basic Solution

$x_i = 0$ if $i \neq j_t$ for some $t \in \{1, \dots$

If $\underline{x}_{\mathcal{B}} \geq \underline{0}$, then $x_{\mathcal{B}}$ is basic feasible solution.

Take the following at face value:

Thm: If an optimal solution exists for an LP in standard form, then there exists a basic feasible solution (bfs) at optimal cost.

Thm: Every optimal solution of an LP in standard form is a convex combination of the optimal basic feasible solutions.

Ex.

General Form

$$\begin{aligned} \max\ & +x_1 + x_2 \\ \text{s.t. } & x_1 \le 3 \\ & x_2 \le 2 \\ & x_1 + x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

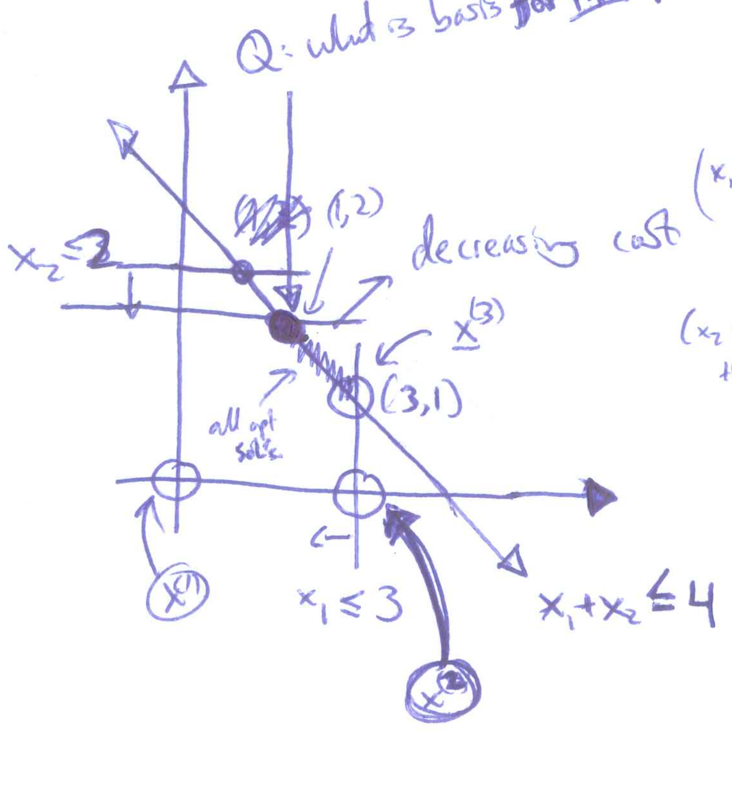
Standard form

$$\begin{aligned} \min\ & -x_1 - x_2 \\ \text{s.t. } & x_1 + x_3 = 3 \\ & x_2 + x_4 = 2 \\ & x_1 + x_2 + x_5 = 4 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$\mathcal{B}$?

Q: what is basis for this pt?



$\mathcal{B}_1 = \{A_3, A_4, A_5\}$ Basic Vars $= \{x_3, x_4, x_5\}$

$\underline{x}^{(1)} = (0, 0, 3, 2, 4)$

(x_1 is better than x_3) $\mathcal{B}_2 = \{A_1, A_4, A_5\}$ Basic Vars $= \{x_1, x_4, x_5\}$

$\underline{x}^{(2)} = (3, 0, 0, 2, 1)$

(x_2 is better than x_5) $\mathcal{B}_3 = \{A_1, A_4, A_2\}$ Basic Vars $= \{x_1, x_4, x_2\}$

$\underline{x}^{(3)} = (3, 1, 0, 1, 0)$

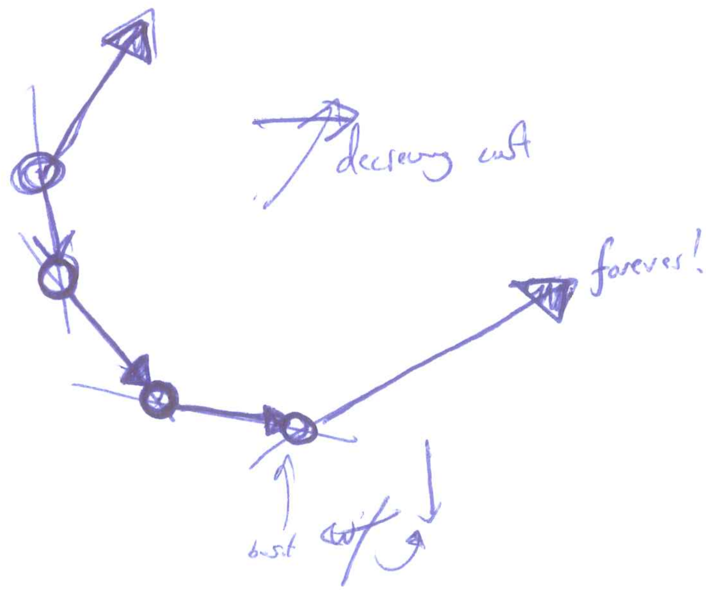
High-Level View of Simplex

The Simplex algorithm finds an optimal bfs through the following process:

1. Find a ~~feasible~~ bfs. (P is feasible iff Q (which has 0's feasible) has optimal = 0.)
↳ If cannot, then infeasible!

2. Improve that bfs
by replacing some base variable with a "better" one.

3. – If no "better" variable, then optimal. (relative cost)
– If we can arbitrarily improve after solution, then unbounded.



Discussed in Appendix.

Things to know:

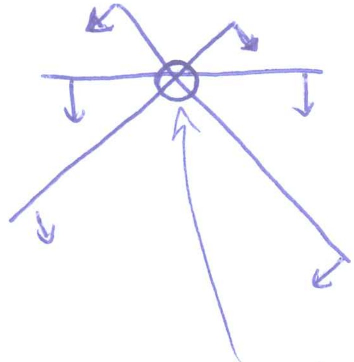
1. Could take exponential time ($2^{f(n,m)}$)
2. Usually is very fast
3. Interior point methods are polynomial, but not as practical.

Using a bfs makes things **easier**.

Beware of <u>Degeneracy</u>:

If a basic variable has value 0, then the bfs is degenerate.

Ex:



3 constraints
2 regular variables,
3 <u>slack</u> variables

three slack variables are zero, but one <u>would</u> be basic!

In this case, changing bases does NOT change <u>solutions</u>.