Duality

Goal: Provide a short proof that a solution is optimal.

Idea: Find as large of a lower bound as possible!

Use: a linear combination of the constraints in the LP.

Ex:

$$
\begin{array}{rrrrrl}
\min & x_1 & +3x_2 & -x_3 & & \\
\text{s.t.} & x_1 & +x_2 & +x_3 & & = 10 \\
 & & & x_3 & -x_4 & \geq 6 \\
 & x_1 & & & +2x_4 & \leq 7 \\
 & & \cancel{x_2} & \cancel{+x_3} & & = \\
 & x_1, & x_2, & \cancel{x_3} & & \geq 0 \\
 & & \cancel{x_2} & x_3 & & \leq 0 \\
 & & & & x_4 & \text{free}
\end{array}
$$

first: standard form

then this general form.

Standard:

Opt $= (0, \frac{1}{2}, \frac{19}{2}, \frac{7}{2})$

Value $= -8$

Want to find ~~function~~ value: $w \leq x_1 + 3x_2 - x_3$

where w is a linear combination of the left hand sides.

Standard Form: $\left(\begin{array}{l}\min\ \underline{c}^T\underline{x} \\ \text{s.t.}\ A\underline{x}=\underline{b} \\ \quad\ \underline{x}\geq\underline{0}.\end{array}\right)$

Solution:

$\underline{x} = \left(0, \frac{1}{2}, \frac{19}{2}, \frac{7}{2}\right)$ val $= -8$

$\underline{y} = (3, -4, -2)$ val $= -8$

$$\min \quad x_1 + 3x_2 - x_3$$

$$\begin{array}{lll}
\text{s.t. } y_1(x_1 + x_2 + x_3) & = (10) & y_1 \\
y_2(x_3 - x_4) & = (6) & y_2 \\
y_3(x_1 + 2x_4) & = (7) & y_3 \\
(x_1, x_2, x_3, x_4 & \geq 0 &
\end{array}$$

Want: Maximize RHS, subject to sum of LHS is at most OPT. (For all $\underline{x}\geq\underline{0}$)

Dual LP.

$$\begin{array}{ll}
\max & 10y_1 + 6y_2 + 7y_3 \quad Q? \\
& y_1 + y_3 \leq 1 \\
& y_1 \leq 3 \\
& y_1 + y_2 \leq -1 \\
& -y_2 + 2y_3 \leq 0 \\
& y_1, \ y_2, \ y_3 \ \underline{\text{free}} \quad Q?
\end{array}$$

collect by x_i

$$\begin{array}{l}
y_1x_1 + y_1x_2 + y_1x_3 \\
+ y_2x_3 - y_2x_4 \\
+ y_3x_1 + 2y_3x_4 \\
\leq x_1 + 3x_2 - x_3.
\end{array}$$

Inequalities: $y_1 + y_3 \leq 1$ since $y_1x_1 + y_3x_1 = \underbrace{(y_1+y_3)x_1 \leq 1x_1}$ for all $\boxed{x_1 \geq 0}$

Variables: y_1, y_2, y_3 are $\underline{\text{free}}$, since

$$y_j(a_j^T\underline{x}) = y_j b_j \quad \text{for } \underline{\text{all}}\ \underline{x} \text{ w/ } A_j\underline{x} = b_j$$

j^{th} row of A

Def: Given an LP in Standard Form:

$$\begin{array}{ll}
\min & \underline{c}^T\underline{x} \\
\text{s.t.} & A\underline{x} = \underline{b} \\
& \underline{x} \geq \underline{0}
\end{array}
\quad \xrightarrow{\text{Dual}} \quad
\begin{array}{ll}
\max & \underline{b}^T\underline{y} \\
\text{s.t.} & A^T\underline{y} \leq \underline{c} \\
& \underline{y} \text{ free}
\end{array}$$

A is $m\times n$, $\underline{x}$ is n-vect, $\underline{c}\in\mathbb{R}^n$, $\underline{b}\in\mathbb{R}^n$

Canonical Form:

$$\min \; x_1 + 3x_2 - x_3$$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 + x_3 &\geq 10 \\ x_3 - x_4 &\geq 6 \\ x_1 + 2x_4 &\geq 7 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Solution:

Primal is <u>unbounded</u>

Dual is <u>infeasible</u>

Want: Maximize RHS, subject to sum of LHS is at most OPT.

Issue: $y_1(x_1 + x_2 + x_3) \geq y_1 \cdot 10$ for <u>all</u> $\underline{x}$

only if $\underline{y_1 \geq 0}$

Dual:

$$\max \; 10y_1 + 6y_2 + 7y_3$$

$$\text{s.t.} \quad \begin{aligned} y_1 + y_3 &\leq 1 \\ y_1 &\leq 3 \\ y_1 + y_2 &\leq -1 \\ -y_2 + 2y_3 &\leq 0 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

General Form:

$$\min \quad x_1 + 3x_2 - x_3$$

$$\begin{aligned}
\text{s.t.} \quad x_1 + x_2 + x_3 &= 10 \\
x_3 - x_4 &\geq 6 \\
x_1 + 2x_4 &\leq 7 \\
x_1, \; x_2 &\geq 0 \\
x_3 &\leq 0 \\
x_4 &\text{ free}
\end{aligned}$$

Solution.

$x = (10, 0, 0, -6)$ value: 10

$y = (1, 0, 0)$ value: 10.

(Issue: $y_1(x_1 + 2x_4) \geq y_1(7)$ only when $y_1 \leq 0$)

Dual:

$$\max \quad 10y_1 + 6y_2 + 7y_3$$

$$\begin{aligned}
\text{s.t.} \quad y_1 + y_3 &\leq 1 \\
y_1 &\leq 3 \\
y_1 + y_2 &\geq -1 \\
-y_2 + 2y_3 &= -1 \\
y_1 &\text{ free} \\
y_2 &\geq 0 \\
y_3 &\leq 0
\end{aligned}$$

≤ 1, ≤ 3 ← x_1 & x_2 are ≥ 0

≥ -1 ← x_3 is ≤ 0

$= -1$ ← x_4 is free

free ← constraint 1 is =

≥ 0 ← constraint 2 is $\geq$

≤ 0 ← constraint 3 is $\leq$

~~Primal~~

Sensible	$x_i \geq 0$,	$a_j^T x \geq b_j$	$y_j \geq 0$	$A_i y \leq c_i$
Odd	x_i free,	$a_j^T x = b_j$	y_j free	$A_i y = c_i$
Bizarre	$x_i \leq 0$	$a_j^T x \leq b_j$	$y_j \leq 0$	$A_i y \geq c_i$
(Vacuous)	$x_i = 0$	$a_j^T x$ unconstrained	$y_j = 0$	$A_i y$ free
	Vars	constraints	Vars	Constraints

$\min \; c^T x$ $\qquad$ $\max \; b^T y$

Thm (Weak Duality): Let $\underline{x}$ be a primal feasible point and $\underline{y}$ a dual feasible point.

$$\underline{c}^T \underline{x} \geq \underline{b}^T \underline{y}$$

Pf: (For primal in Standard Form)

since primal is $\begin{aligned} &\min \underline{c}^T \underline{x} \\ \text{s.t. } &A\underline{x} = \underline{b} \\ &\underline{x} \geq 0 \end{aligned}$ and dual is $\begin{aligned} &\max \underline{b}^T \underline{y} \\ &A^T \underline{y} \leq \underline{c} \\ &\underline{y} \geq \underline{0}, \end{aligned}$

we have

$$\underline{b}^T \underline{y} = \underline{y}^T \underline{b} = \underline{y}^T (A\underline{x}) = (\underline{y}^T A)\underline{x} = (A^T \underline{y})^T \underline{x} \leq \underline{c}^T \underline{x}.$$

$\square$

We have the following pairs of end results for Primal/Dual pairs.

Primal \ Dual	Opt	Infeas.	Unbound.
Opt	✓	✗	✗
Infeas.	✗	✓	✓
Unbound.	✗	✓	✗

Thm: The Dual of the Dual is (equivalent to) the Primal.

Diet Problem:

1. Need certain vitamins.
2. Food provides certain nutrients.
3. Want to minimize <u>cost</u> of food.

Let $F_1, F_2, \dots, F_n$ be n food items.

Let $V_1, \dots, V_m$ be m vitamins.

Let $a_{i,j}$ be the amount of vitamin V_i in one serving of F_j. $\left[A = [a_{i,j}]_{m \times n}\right.$

Let c_j be the <u>cost</u> of a serving of F_j.

Let x_j be <u>the amount</u> of F_j I will eat. (in # of servings).

Let b_i be the amount of V_i <u>required</u>.

Then, my diet is determined by

$$\begin{aligned} \min\ & \underline{c}^T \underline{x} \\ \text{s.t.}\ & A\underline{x} \geq \underline{b} \leftarrow \text{meets } \underline{\text{required needs}} \\ & \underline{x} \geq 0 \leftarrow \text{not barfing} \end{aligned}$$

A <u>vitamin company</u> wants to provide <u>supplements</u> for $V_1, \dots, V_m$.

Let y_i be the amount the company <u>charges</u> for V_i.

Wants: You to buy these vitamins instead of any foodstuff.

("Bread" is equivalent to 5g fiber, 3g protein, 15g sugar, etc.)

They want to make as much money as possible!

$$\begin{aligned} \max\ & \underline{b}^T \underline{y} \\ \text{s.t.}\ & A^T \underline{y} \leq \underline{c} \\ & \underline{y} \geq \underline{0} \leftarrow \text{not } \underline{\text{paying you}} \text{ to take the vitamins!} \end{aligned}$$

Shortest Paths and its dual

(See Implementation Report ① for more details.)

$$\min \sum_{ij \in E} \omega(i,j)\, x_{i,j}$$

s.t.

$$\sum_{i:\, is \in E} \cancel{\omega(\ldots)}\, x_{i,s} - \sum_{i:\, si \in E} \cancel{\omega(s,i)}\, x_{s,i} \quad \cancel{(=-1)} \qquad (s)$$

Sum of these equals (s) constraint!

$$\sum_{i:\, ij \in E} \cancel{\ldots}\, x_{i,j} - \sum_{i:\, ji \in E} \cancel{\ldots}\, x_{j,i} = 0 \qquad (j) \in \{2,\ldots,n-1\}$$

$$\underbrace{\sum_{i:\, it \in E} \cancel{\ldots}\, x_{i,t}}_{\text{incoming}} - \underbrace{\sum_{i:\, ti \in E} \cancel{\ldots}\, x_{t,i}}_{\text{outgoing}} = 1 \qquad (t)$$

$$x_{i,j} \geq 0.$$

Dual:

$$\max \quad y_t \; (-y_s)$$

s.t.

$$y_i - y_j \leq w(i,j) \qquad (i,j)$$

$$y_s = 0$$

$$y_2, \ldots, y_{n-1}, y_t \quad \underline{\underline{\text{free}}}$$

Interpretation: <u>Stretch</u> the graph on the real line, trying to separate s and t, but the edges keep us from succeeding!