Strong Duality

Recap:

Primal Problem: minimize $\underline{c}^T \underline{x}$
s.t. constraints $A\underline{x} \; ? \; \underline{b}$
(ie. $A\underline{x} = \underline{b}$, $Ax \geq b$)
and $\underline{x} \geq 0$ (or others.)

Dual Problem: maximize $\underline{b}^T \underline{y}$
s.t. constraints that imply $\underline{b}^T \underline{y} \leq \underline{c}^T \underline{x}$
for all feasible $\underline{x}$.

Weak Duality: $\underline{b}^T \underline{y} \leq \underline{c}^T \underline{x}$ for all primal feasible $\underline{x}$
and dual feasible $\underline{y}$.

Strong Duality: If $\underline{y}^*$ is ~~primal~~ dual optimal, ~~[illegible]~~,
then there exists ~~dual~~ primal feasible $\underline{x}^*$ (also ~~dual~~ primal optimal) with
$$\left(\text{for all dual feasible } \underline{y} \;\; \underline{b}^T \underline{y} \leq \right) \underline{b}^T \underline{y}^* = \underline{c}^T \underline{x}^* \left(\leq \underline{c}^T \underline{x} \text{ for all primal feasible } \underline{x} \right)$$

Pf of Strong Duality by way of Farkas' Lemma.

<u>Farkas' Lemma</u>: Let A be an $m \times n$ matrix and $\underline{b}$ an m-vector.
Then, exactly <u>one</u> of the following holds:

1. There exists some $\underline{x} \geq \underline{0}$ s.t. $A\underline{x} = \underline{b}$.
2. There exists some $\underline{y}$ s.th $A^T \underline{y} \geq \underline{0}$ and $\underline{b}^T \underline{y} < 0$.

HW. Prove Farkas' Lemma from Strong Duality.

Cor: Suppose that the system $A^T \underline{y} \le \underline{c}$ has at least one solution, and let δ be a number. Then,

every feasible solution to the system $A^T\underline{y} \le \underline{c}$ satisfies $\underline{b}^T\underline{y} \le \delta$

iff there exists some $\underline{x} \ge \underline{0}$ s/t $A\underline{x} = \underline{b}$ and $\underline{c}^T\underline{x} \le \delta$.

Pf: Consider the pair of problems

$$\begin{array}{lllll} \max & \underline{b}^T\underline{y} & \underset{\text{dual}}{\overset{\text{take}}{\longleftarrow}} & \min & \underline{c}^T\underline{x} \\ \text{s.t.} & A^T\underline{y} \le \underline{c} & & & A\underline{x} = \underline{b} \\ & \underline{y} \text{ free} & & & \underline{x} \ge \underline{0}. \end{array}$$

($\Leftarrow$) by weak duality.

($\Rightarrow$) Suppose no $\underline{x} \ge \underline{0}$ w/ $A\underline{x} = \underline{b}$ satisfies $\underline{c}^T\underline{x} \le \delta$.

Then, there exists no $\underline{x} \ge \underline{0}$, $\lambda \ge 0$ s/t

$$\left[\begin{array}{c|c} A & 0 \\ \hline c^T & 1 \end{array}\right] \begin{pmatrix} \underline{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} \underline{b} \\ \delta \end{pmatrix}.$$

By the Farkas' Lemma, we then have some $\underline{z}, w$ s/t

$$\left[\begin{array}{c|c} A^T & \underline{c} \\ \hline \underline{0}^T & 1 \end{array}\right] \begin{pmatrix} \underline{z} \\ w \end{pmatrix} \ge \underline{0} \quad \text{and} \quad (\underline{b}^T \ \delta)\begin{pmatrix} \underline{z} \\ w \end{pmatrix} < 0.$$

If $w = 0$, then $A^T\underline{z} \ge \underline{0}$ and $\underline{b}^T\underline{z} < 0$.

We assumed there exists $\underline{y}$ with $A^T\underline{y} \le \underline{c}$, so for all $\gamma > 0$,

$$A^T(\underline{y} - \gamma\underline{z}) \le \underline{c} \quad \text{and} \quad \underline{\underline{b}^T(\underline{y} - \gamma\underline{z}) \to +\infty} \ \ \underline{\underline{!!!}}$$

This contradicts the fact that $\underline{b}^T(\underline{y} - \gamma\underline{z}) \le \delta$ (supposedly!) $\lightning$

Now, suppose $w \neq 0$, so

$$A^T \underline{z} + \underline{c} \cdot w \geq \underline{0}, \quad \text{and} \quad 1w \geq 0 \Rightarrow \underline{\underline{w \geq 0}}$$

$$\Rightarrow A^T \underline{z} \geq (-w)\, \underline{c}$$

$$\Rightarrow A^T\left(\frac{-1}{w} z\right) \leq \underline{c}$$

Also: $b^T z + \delta w < 0$

so $b^T z < -\delta w$

Let $y^* = \frac{-1}{w} \underline{z}$, we have $A^T y \leq \underline{c}$ and

$$b^T y = b^T\left(\frac{-1}{w} z\right) < +\delta$$

$$b^T z < -\delta w$$

$$\Rightarrow b^T\left(\frac{-1}{w} z\right) > \delta$$

~~so, $b^T y$~~

This contradicts that $b^T y \leq \delta$!!! $\blacksquare$

Now, let us consider primal & dual LPs.

Primal is equivalent to:

$$\begin{aligned} \min \quad & \underline{c}^T \underline{x} \\ \text{s.t.} \quad & A \underline{x} = \underline{b} \\ & \underline{x} \geq 0, \end{aligned}$$

Dual is equivalent to:

$$\begin{aligned} \max \quad & \underline{b}^T y \\ \text{s.t.} \quad & A^T y \leq \underline{c} \\ & y \text{ free.} \end{aligned}$$

If y^* is a ~~primal~~ dual optimal, then let $\delta = \underline{b}^T y^*$ and we have $b^T y \leq \delta$ for all feasible y.

By Cor, $\exists\ \underline{x}^* \geq \underline{0}$ with $A\underline{x}^* = \underline{b}$ and $c^T \underline{x}^* \leq \delta$.

Since $b^T y^* \leq c^T \underline{x}^* \leq \delta = b^T y^*$, equality holds and $\underline{x}^*$ is primal opt! $\blacksquare$

Chapter 3. Maximum Flow Problems

A company owns factories $F_1, \dots, F_n$ that each make the same item, which is sold at stores $S_1, \dots, S_m$.

The holiday's newest toy!

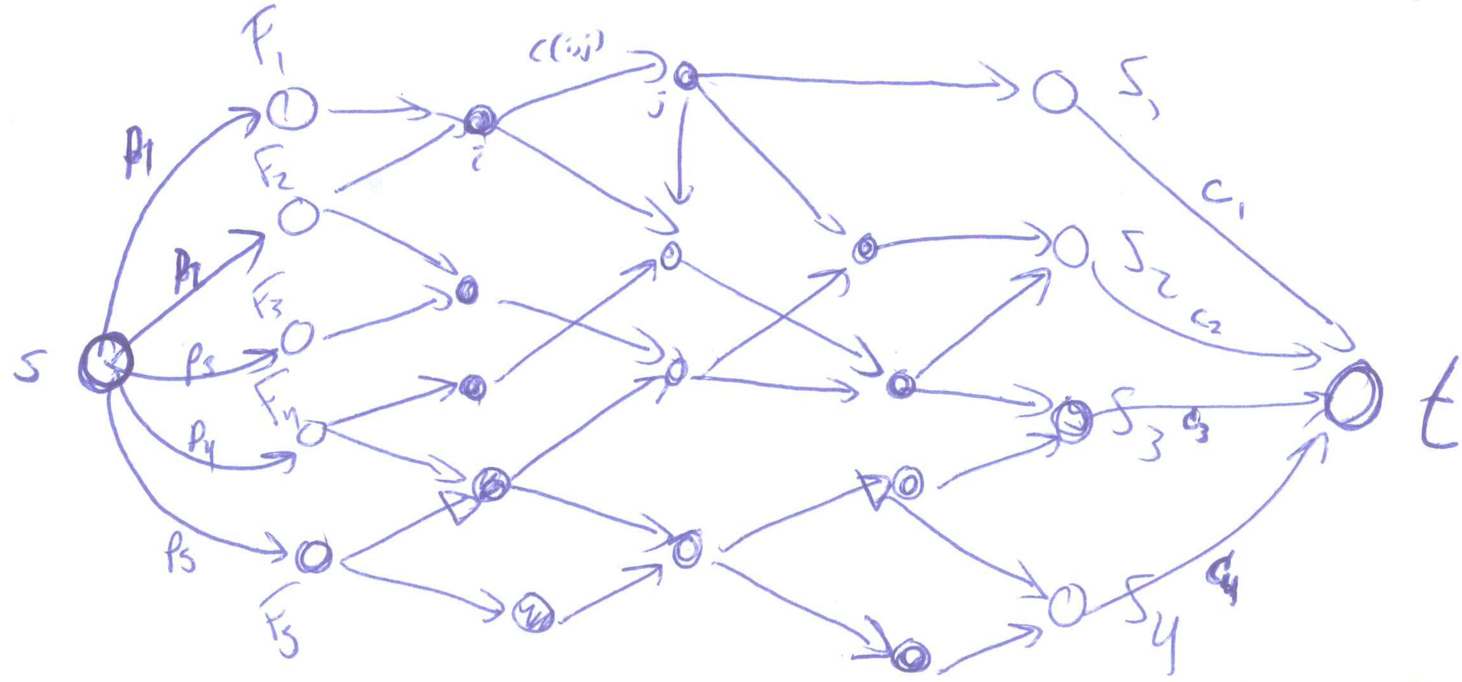
Problem: Factory F_i can only make p_i toys per day,

Store S_j can only sell c_j toys per day,

AND we need to get the toys to the stores!

Every route has a finite capacity:

City i to city j has capacity $c(i,j)$ toys per day.



Want to maximize # of toys sold without saturating the market, overtaxing a factory, or overloading a shipping route.

Other Applications:

Internet Streaming
Water Flow
Traffic?
Trains, Planes, ...

Focus on Trucks hauling toys. From $\underset{\text{source}}{r}$ to $\underset{\text{sink}}{s}$. Each truck T_ℓ is given a route, i.e. an r,s-path P_ℓ, to follow. Let $P_1, \dots, P_t$ be a list of routes for t trucks.

Every vertex $v \notin \{r, s\}$ has the same # of incoming & outgoing edges. So, if we let x_{uv} = # of paths using edge $uv \in E$, then

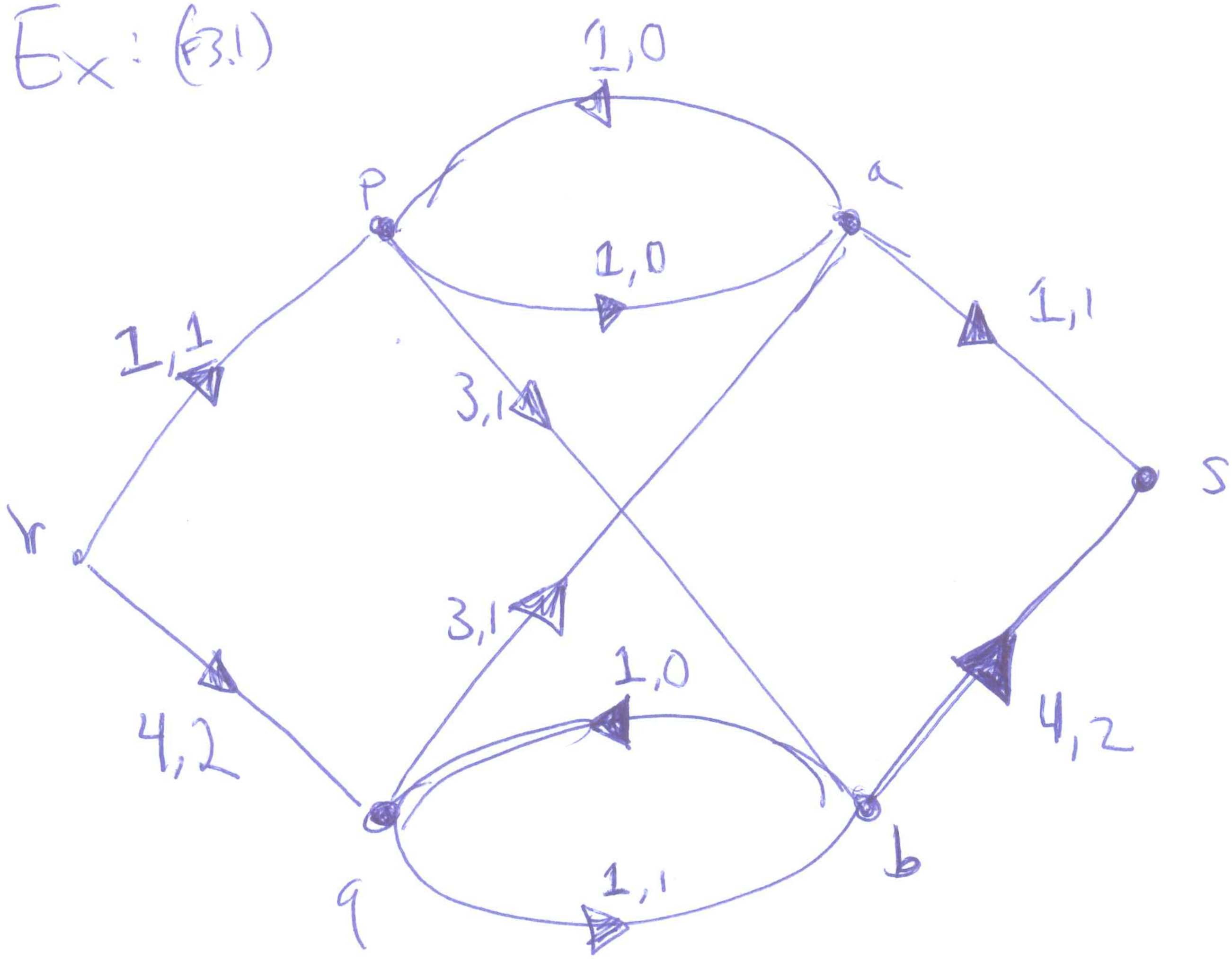
$$\text{(Conservation)} \quad \underbrace{\sum_{u: uv \in E} x_{u,v} - \sum_{w: vw \in E} x_{v,w}}_{f_x(v)} = 0 \quad (v \notin \{r, s\})$$

Also every edge uv has a capacity $c_{(u,v)} \geq 0$

If we are sending discrete trucks, then

$x_{u,v} \geq 0$ is integer.

Ex: (F3.1)



$P_1 =$ r p b s

$P_2 =$ r q b s

$P_3 =$ r q a s

Path Packing Problem:

Maximize t such that t- rs-paths $P_1, \dots, P_t$ exist such that every edge $x_{u,v}$ is used in at most $c(u,v)$ of them.

Integral Max Flows Problem:

?

gives a solution.

$$\max \quad f_{\underline{x}}(s)$$

$$\text{s.t.} \quad f_{\underline{x}}(v) = 0 \qquad v \notin \{r, s\}$$

$$0 \le x_{u,v} \le c(u,v) \qquad uv \in E(G)$$

$$x_{u,v} \qquad \underline{\text{integer}}.$$

Feasible solution: an (integral) feasible (r,s)-flow of value $f_{\underline{x}}(s)$

Prop. 31: There exists a family $(P_1, \dots, P_t)$ of rs-paths such that $|\{i : P_i \text{ uses } e\}| \le c(e)$ for all $e \in E$ if and only if there exists an <u>integral feasible (r,s)-flow</u> of <u>value</u> $\underline{t}$.

$$\max \sum_{rs\text{-paths } P_j} x_{P_j}$$

$$\text{s.t.} \sum_{P_j \ni e} x_{P_j} \le c(e) \quad \forall e$$

$$x_{P_j} \ge 0, \quad \text{integer}.$$

Pf of 3.1: (⇒) Assign $x_e = \sum_{P_j \ni e} x_{P_j}$. ✓

(⇐) ~~If there exists a path~~, Strengthened Induction: $\exists$ a path packing w/ $\sum_{P_j \ni e} x_{P_j} \le x_e$

Induct on $t \ge 0$, secondarily on $\sum x_e$

If $t = 0$, then assigning $x_{P_j} = 0$ also has value 0.

Now, let $t > 0$. We must show that some rs-path P_j has

$$\min\{x_e : e \in P_j\} \ge 1.$$

Since $val(f_{\underline{x}}) = t$, we have

$$t = f_{\underline{x}}(s) = \sum_{v:\, vs \in E} x_{v,s} - \sum_{w:\, sw \in E} x_{s,w} = t$$

We will ~~walk~~ backward along edges with positive value, until reaching either r (and we have an rs-path) or a vertex already in the walk.

We will terminate in this way, due to conservation constraints and G is finite.

1. We reach r. Then our walk (in reverse) is an rs-path P_j.

   Assign $x_{P_j} = \min\{x_e : e \in P_j\}$.

   Since all e's we used had positive, integer value, x_{P_j} is positive, integer value.

   Define $x'_e = \begin{cases} x_e - x_{P_j} & e \in P_j \\ x_e & \text{otherwise.} \end{cases}$

   $\underline{x}'$ is an integer feasible flow w/ $f_{\underline{x}'}(r) = t - x_{P_j} < t$.

   Induct to find a path packing of value $t - x_{P_j}$, and x_{P_j} to get a path packing of value t.

   True by strengthened induction! (Capacities are not violated)

2. If we reach a vertex v already in our walk, then we have a cycle!

Remove value from the cycle. $\square$

Skolem: Every rs-flow of nonnegative value is the sum of at most $m = |E|$ flows, each of which is a path flow, or a circuit flow (cycle flow).

Duality (Combinatorial Perspective)

"Duality is how competing entities find an equilibrium solution."

Max Flow: A terrorist communication network is modeled as a graph, where the leader sends orders down through a network. He wants to send as much information, but certain links are less secure than others. More secure ≡ Higher Capacity.

~~M~~ Dual : The NSA has wire tapped to discover the entire communication network, and wants to shut it down. It wants to remove communication links subject to

More Secure ≡ Higher Cost to Shut Down.

Min Cut: $\min \sum_{e \in E} c(e)\, y_e$

s.t.

$$\sum_{e \in P_j} y_e \geq 1$$

$$y_e \geq 0 \text{ (integer)}$$