Breadth-First-Search will find ~~the~~ shortest path
(in terms of fewest arcs)

Thm 3.10 (Dinitz; Edmonds and Karp): If each augmentation of the augmenting paths algorithm is a shortest path, then there are at most nm augmentations.

$|V(G)| \cdot |E(G)|$

Cor: The augmenting path alg. w/ BFS ~~requires~~ solves the max flow problem in $O(nm^2)$ time.

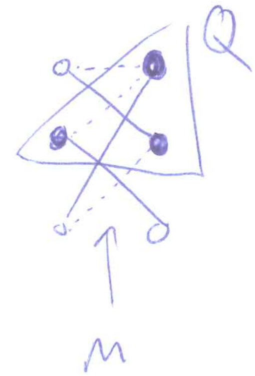
Note: $O(nm^2) = O(n^5)$.
This can be simplified if we restrict our graph.

If G is planar, then $m \leq 3n-6$, so $O(n^3)$.
(We can do better, using different algorithms.)

Emphasize Integrality!!!

Applications

Bipartite Matchings + Covers



Def: G bipartite, matching^M, vertex cover^Q

Prop: For a matching M and a vertex cover Q,

$$|M| \leq |Q|.$$

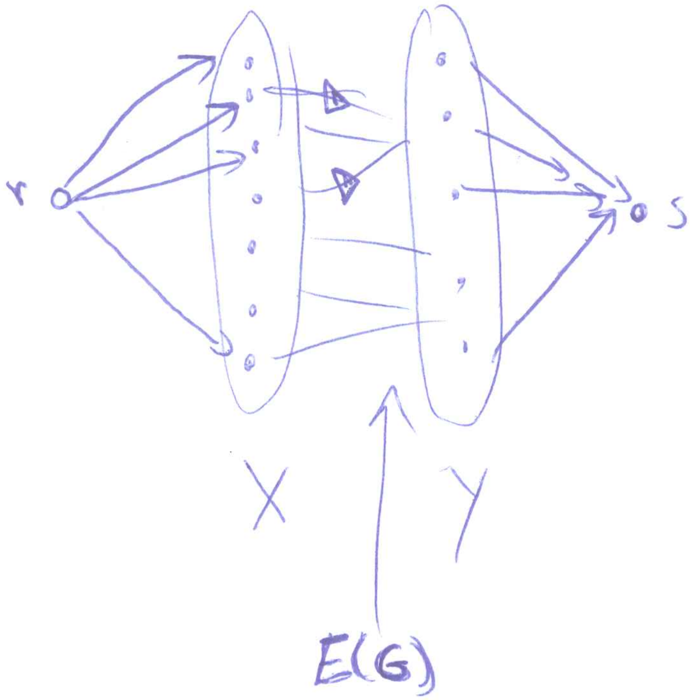
Pf: Every edge in M is covered by something in Q.
These are distinct since edges in M are disjoint.

Thm 3.14 (König's Theorem): For a bipartite graph G,

$$\max\{|M| : M \text{ a matching}\} = \min\{|C| : C \text{ a cover}\}.$$

Pf (from Max-Flow/Min-Cut):

Let G be bipartite w/ bipartition X, Y.



Direct the edges from X to Y. Place capacity ∞ on these edges.

Place r, s as new vertices.

$r \xrightarrow{1} v \quad \forall v \in X$

$u \xrightarrow{1} s \quad \forall u \in Y$.

Find an <u>integral</u> maximum flow x in this flow network.

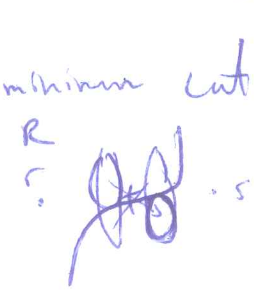
Since every vertex $v \in X$ has an incoming capacity of 1, its incoming flow is either 0 or 1.

So, it either has 0 or 1 outgoing edges w/ flow on them.

Similarly, all $u \in Y$ have outgoing capacity of 1, so incoming is 0 or 1.

Thus, let $M = \{e \in E(G) : x_e > 0\}$. M is a matching.

Let $\delta(R)$ be a minimum cut in the flow network.



Since max flow = min cut, the cut has finite capacity, so no $e \in E(G) \in \delta(R)$.

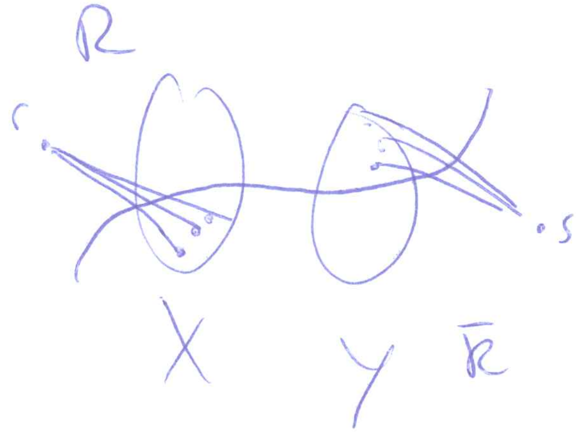
Let $Q = (\bar{R} \cap X) \cup (R \cap Y)$. Q is a cover.

$$|Q| = |\bar{R} \cap X| + |R \cap Y|.$$

Flow ~~from~~ R to $\bar{R}$ is <u>full</u>, so

for all $v \in \bar{R} \cap X$, we have $r \to v$ is full.

For all $u \in R \cap Y$, $u \to s$ is full.



<u>But</u> no flow goes from $\bar{R}$ to R, so for all $v \in X \cap \bar{R}$, $u \in Y \cap R$, no flow passes from v to u.

Thus, in M, no $v \in X \cap \bar{R}$ is matched to any $u \in Y \cap R$, but all vertices in Q <u>are</u> matched!

Thus, $|Q| = |M|$. $\square$

Optimal Closure in a Digraph

Let G be a digraph whose

vertices are "projects", with a given <u>value</u> (could be positive or negative)

(do you <u>break even</u>?)

edges are <u>dependencies</u> (if we do x, then we must buy "first")

We want to find a set $A \subseteq V(G)$ s.t. all dependencies are fullfilled and the value is <u>maximized</u>.

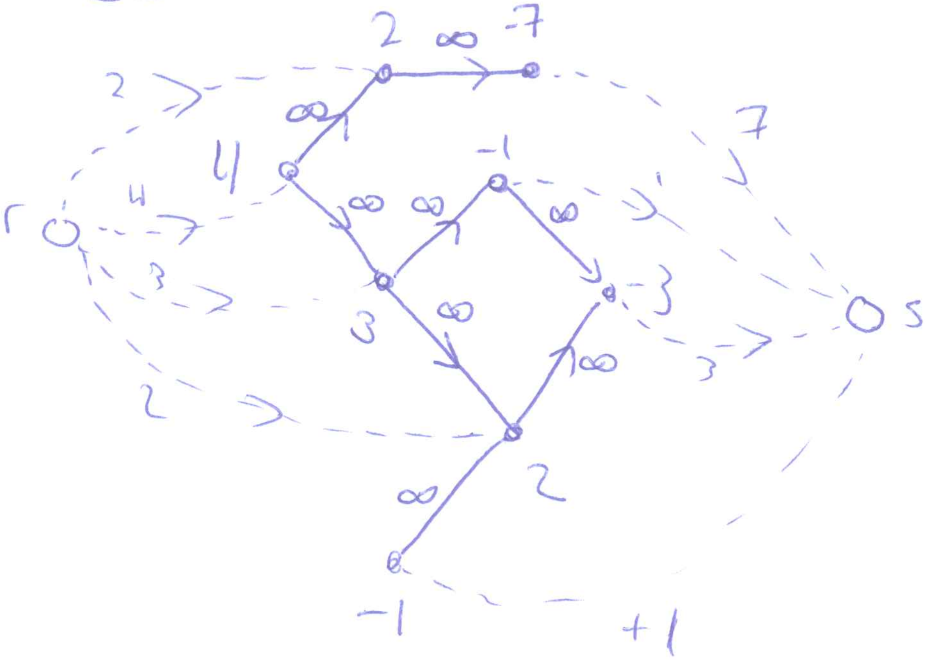
Application: Mining, need to excavate "above" before mining down.

Build a flow network: Start w/ G and ∞-capacities.

for all $v \in V(G)$ with $b_v \geq 0$, add an edge $r \xrightarrow{b_v} v$

for all $v \in V(G)$ with $b_v < 0$, add an edge $v \xrightarrow{-b_v} s$

Ex:



For ~~a minimum~~ cut $\delta(R)$, let $A = R \cap V(G)$ $(R - r)$.

Also, $\delta(R) \cap E(G) = \emptyset$ by ∞ cap's.

So, $c(\delta(R)) = \sum_{v \notin R:\, b_v > 0} b_v - \sum_{v \in R:\, b_v < 0} b_v$

If we ~~subtract~~ add $\sum_{v:\, b_v \geq 0} b_v$ to both ~~sides~~ terms, we have

$\cancel{c(\delta(R)) +}$

$\cancel{c(\delta(R)) + \sum_{v \notin A,\, b_v \geq 0} b_v = \sum_{v \in V:\, b_v \geq 0} b_v - \sum_{v \in A,\, b_v < 0} b_v + \sum_{v \in A,\, b_v \geq 0} b_v}$

double-counted!

constant

maximized!

$$c(\delta(R)) = \sum_{\substack{v \in V \\ b_v \geq 0}} b_v - \left(\sum_{\substack{v \in A \\ b_v < 0}} b_v + \sum_{\substack{v \in A \\ b_v \geq 0}} b_v \right) = \sum_{\substack{v \in V \\ b_v \geq 0}} b_v - b(A)$$

Flow Feasibility Problems

Consider transportation model: $P \cup Q$, P's are factories, Q's are stores.

edge pq is for all delivery options.

Store q can sell at most a_q items.

The factory p will build b_p items.

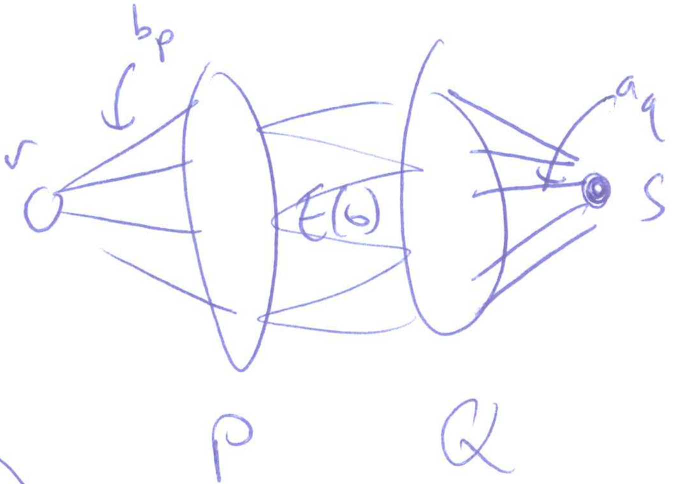
So, not a max flow, but a feasible flow.

$$\forall q, \quad \sum_{p:\, pq \in E} x_{p,q} \le a_q$$

$$\forall p \quad \sum_{q:\, pq \in E} x_{p,q} = b_p .$$

$x_{p,q} \ge 0$, integral.

Model as flow:



Q: Is there a flow of value $\sum_{p \in P} b_p$?

A more general problem: (a_v, b_v, l_e, u_e) ~~integer~~ integer

$$a_v \le f_x(v) \le b_v \quad \forall v \in V,$$
$$l_e \le x_e \le u_e \quad \forall e \in E.$$

Note: If $l_e = 0$ and $a_v = b_v$ always, then we can solve!

$\rightarrow$ Reduce it!

Consider $x'_e = x_e - l_e \ge 0$, integer.

Then, $$f_x(v) = \sum_{u: uv \in E} x_{u,v} - \sum_{w: vw \in E} x_{v,w}$$

$$= \sum_{u: uv\in E} (x'_{u,v} + l_{u,v}) - \sum_{w: vw \in E} (x'_{v,w} + l_{v,w})$$

$$= \cancel{\sum_{u:uv\in E} x_{u,v}}\ f_{x'}(v) + \underbrace{\sum_{u: uv\in E} l_{u,v} - \sum_{w: vw\in E} l_{v,w}}_{l(v)}$$

$$a'_v = a_v - l(v)$$
$$b'_v = b_v - l(v)$$

Then $a_v \le f_x(v) \le b_v$ iff $a'_v \le \underset{f_x(v) - l(v)}{\underset{\parallel}{f_{x'}(v)}} \le b'_v$

~~So now, assume $l_e = 0$ $\forall e \in E$.~~

We make the following restriction:

$a_v = b_v$, integer.

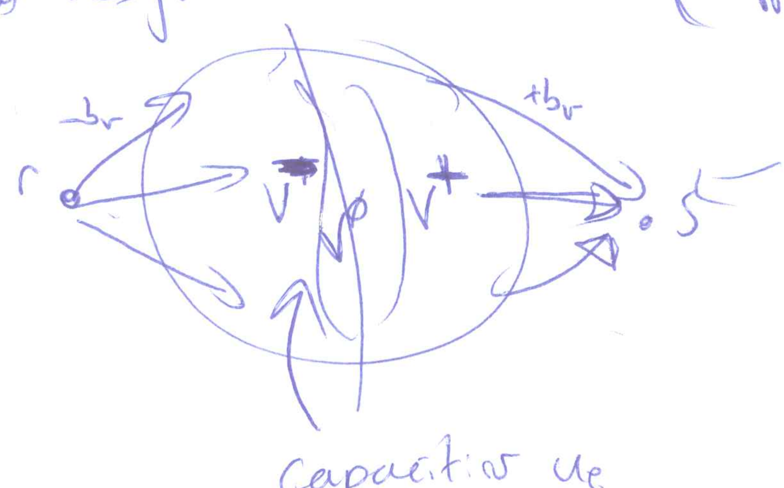
So, $f_{\underline{x}}(v) = b_v$ means $\begin{cases} v \text{ } \underline{\text{consumes}} \text{ } b_v & (b_v > 0) \\ v \text{ conserves flow} & (b_v = 0) \\ \text{or} & \\ v \text{ } \underline{\text{supplies}} \text{ } |b_v| & (b_v < 0) \end{cases}$

~~Case 1: $b_v =$~~

Since $\sum f_{\underline{x}}(v) = 0$, $\sum b_v = 0$ or else infeasible.
(global conservation)

~~If $b_v = 0$ $\forall v$, then a feasible flow is a $\underline{\text{circulation}}$.~~
~~we allow $l_v > 0$~~
~~(no new flow, no consumed flow)~~

Build a flow network G' as: (Suppose $l_e = 0$)



Has a max flow of value $\sum_{v : b_v > 0} b_v$

iff

orig G has a feasible flow.

Thm 3.15. There exists a solution to (*)

$$(*) \quad \begin{aligned} &f_{\underline{x}}(v) = b_v \\ &0 \le x_e \le u_e \end{aligned}$$

iff $\sum b_v = 0$ and for every $A \subseteq V$, $b(A) \le u(\delta(\bar{A}))$.

If $\underline{b}$ and $\underline{u}$ are integral, then (*) has an integer solution iff the same conditions hold.

Pf: Let $\underline{x}$ be a max flow to the new network G'.

$f_{\underline{x}}(v) = b_v$ iff $f_{\underline{x}}(s) = \sum_{v: b_v > 0} b_v$.

By Max Flow/Min-Cut, all rs-cuts R have $A = R - r \subseteq V(G)$ and

$$\sum_{b_v > 0} b_v \le u(\delta_{G'}(A \cup \{r\})) = \underbrace{\sum_{\substack{v \notin A,\\ b_v < 0}} -b_v}_{} + \underbrace{\sum_{\substack{v \in A,\\ b_v > 0}} b_v}_{} + \underline{u(\delta_G(A))}$$

So, $$\sum_{b_v > 0} b_v \le \sum_{\substack{v \notin A\\ b_v < 0}} -b_v + \sum_{\substack{v \in A\\ b_v > 0}} b_v + u(\delta(A))$$

$$\iff \quad u(\delta(A)) \ge \sum_{v \notin A} b_v.$$

If $B = \bar{A}$, then $\underbrace{u(\delta(\bar{B})) \ge \sum_{v \in B} b_v = b(B)}$ $\square$

If $b_v = 0 \ \forall v$, then a feasible flow to

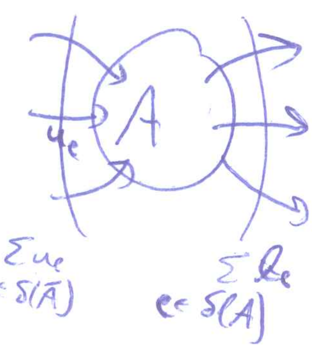
$$f_x(v) = 0$$

$$l_e \le x_e \le u_e$$

is called a <u>circulation</u>.

Thm 3.17 (Hoffman's Circulation Thm):
Given a digraph G, $\underline{l}, \underline{u} \in \mathbb{R}^{E(G)}$, $\underline{l} \le \underline{u}$, then there is a circulation $\underline{x}$ w/ $\underline{l} \le \underline{x} \le \underline{u}$ iff every $A \subseteq V(G)$ satisfies $\underline{u(\delta(\bar{A})) \ge l(\delta(A))}$.

(i.e. amount coming <u>in</u> is <u>enough</u> for how much comes out.)



Also, <u>integrality</u>.

Pf: Perform our translation of $l_e \to 0$, $u_e \to u_e - l_e$, and adjust b_v from 0 to $\sum l_e\, l(v)$.
Then, Thm 3.15 applies. $b(V) = 0$ clearly.

$$u'(\delta(A)) = u(\delta(A)) - l(\delta(A)).$$

If ~~$u(\delta(A)) \ge l(\delta(A)) \ \forall A$, then $u'(\delta(A))$~~ ≥ 0.

There exists a solution iff

$$u(\delta(\bar{A})) - \ell(\delta(\bar{A})) = u'(\delta(\bar{A})) \geq -b'(A)$$

$$= \ell(\delta(A)) - \ell(\delta(\bar{A}))$$

$$\boxed{u(\delta(\bar{A})) \geq \ell(\delta(A))}$$

$\square$