Ch 5: Optimal Matchings

Def: Matching M, M-covered, M-exposed

deficiency^{def}, matching # $^{\nu = \alpha'}$, perfect matching

In bipartite graphs, following reduce to flows:

- Max Matching
- Max Weighted Matching
- Min Weight Perfect Matching ✱
  (min cost flows)

Since we skipped min-cost flows, we will cover this!

Vertex Cover, #, β

Recall: $\alpha' \leq \beta$: for bipartite, $\alpha' = \beta$.

Def: M-alternating paths, M-augmenting paths

Thm 5.1 (Aug. Path Thm for Matchings):

A matching M in a graph $G=(V,E)$ is maximum iff there is no M-augmenting path.

Pf: ($\Rightarrow$) If there is an M-augmenting path, then M is not maximum, by performing augmentation.

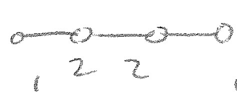
($\Leftarrow$) If M is not maximum, then let M' be a <u>larger</u> matching. $|M|<|M'|$.

The <u>symmetric difference</u> $M \triangle M' = (M-M') \cup (M'-M) = (M\cup M') - (M \cap M')$.

Contains edges in <u>exactly one</u> of M and M'.

Lemma: If M & M' are matchings, then $M \triangle M'$ is a collection of disjoint paths & even cycles. (Each is M-alternating)

Pf: Every vertex is in 0, 1, or 2 edges of $M \triangle M'$.

So,  paths, and  cycles.

Since those of degree 2 have 1 from each, these paths and cycles alternate between M & M'. $\square$

$\Rightarrow$ Cycles are even.

Now, for M, M' w/ $|M|<|M'|$,

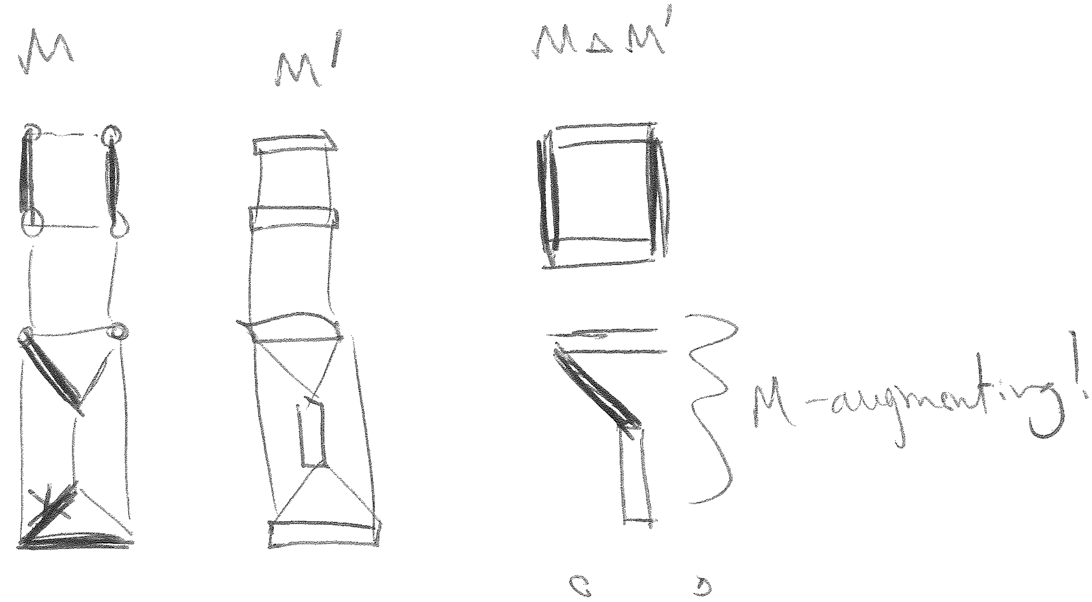
$$|M| = |M-M'| + |M \cap M'|$$

and

$$|M'| = \underbrace{|M'-M|}_{\text{in } M\triangle M'} + \underbrace{|M\cap M'|}_{\text{NOT in } M \triangle M'}$$

<u>over</u> half of edges in $M \triangle M'$ are from M'. Even cycles contribute equal amounts from each M & M', so there must exist a path P in $M \triangle M'$ that starts & ends w/ M'-edges.

P is an M-augmenting path! $\square$



Recall Duality of Vertex Covers.

$\beta(G) = \min\{|S| : S \subseteq V \text{ is a vertex cover}\}$.

In bipartite G, $\alpha'(G) = \beta(G)$.

For any G, $\alpha'(G) \le \beta(G)$.

Consider an odd cycle.

$\alpha' = 2$
$\beta = 3$

$\alpha' = 3$
$\beta = 4$

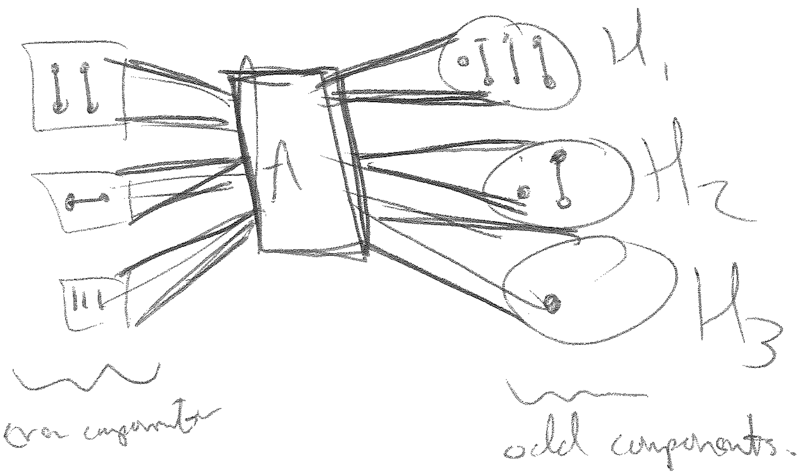
Odd Components

A component is a maximal connected subgraph.

Let G be a graph and $A \subseteq V(G)$.

In $G - A$, let $H_1, \ldots, H_k$ be the odd components
(Some even components may exist.)

In ~~each~~ any matching of G odd component H_i, some vertex ~~cannot~~ is left unmatched.



To find a match for these vertices, they must match to something in A! (Only remaining neighbors are in A).

So, the # of matched edges is ~~at~~ at most

$$|M| \le \frac{1}{2}\left(|V| - \left(\underbrace{oc(G-A)}_{\text{\# of odd components}} - |A|\right)\right)$$

Really:

$$\underbrace{2|M|}_{\substack{\text{\# of matched}\\ \text{vtx's}}} \le \underbrace{|V|}_{\substack{\text{all}\\ \text{vertices}}} - \underbrace{(oc(G-A) - |A|)}_{\substack{\text{vertices in odd components that}\\ \text{cannot match to } A.}}$$

Recall: $def(G) = |V| - 2\alpha'(G)$

So, $def(G) \ge oc(G-A) - |A|$.

Hall's Thm: Let $G=(X\cup Y, E)$ be bipartite.

G has a matching saturating X iff for all sets $S \subseteq X$,

$\left.|S| \le |N(S)|.\right\}$ Hall's Condition.

Pf: $(\Rightarrow)$ Let M saturate X.

$S \subseteq X$, then the vertices matched to S form a subset $T \subseteq N(S)$.

$|S| = |T| \le |N(S)|$.

$(\Leftarrow)$ By strong duality w/ vertex cover.

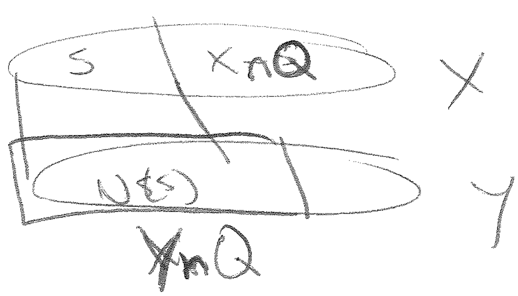
Let Q be a minimum vertex cover.

~~and suppose $|Q| < |X|$, AND~~ $|S| \le |N(S)|$ for all $S \subseteq X$.

Consider $S = X - Q$. This set is nonempty.

Then, $N(S) \subseteq Q$, since all edges incident to S are covered by Q.

Now, let $Q' = X$.



$$\begin{aligned} |Q'| &= |X\cap Q| + |S| \\ &\le |X \cap Q| + |N(S)| \\ &\le |X\cap Q| + |Y \cap Q| \\ &= |Q|. \end{aligned}$$

Thus, the set $Q' = X$ is a minimum vertex cover

Thus, there exists a matching M of size $|M| = |X|$

and therefore M saturates X. $\square$

Thm 5.2 (Tutte-Berge Formula): For a graph $G=(V,E)$, we have

$$\alpha'(G) = \max\{|M| : M \text{ is a matching in } G\} = \min\left\{\tfrac{1}{2}\left(|V(G)| - (oc(G-A) - |A|\right) : A \subseteq V\right\}$$

Equivalently,

$$\operatorname{def}(G) = \max\{oc(G-A) - |A| : A \subseteq V\}$$

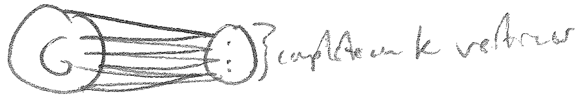
Cor (Tutte's Thm): G has a perfect matching iff $|A| \geq oc(G-A)$ $\forall A \subseteq V$.

Pf: ~~We will show the Tutte-Berge formula by induction~~

Instead of proving the Berge-Tutte Formula, we will show it as a corollary of Tutte's Theorem.

Pf: Let G be a graph with ~~matching number $\alpha'(G) = \frac{n}{2}+k$~~ ~~deficiency $\operatorname{def}(G) = k$~~. $k = \max\{oc(G-A) - |A| : A \subseteq V\}$

Consider $G' = G \vee K_k$:



~~G has $\alpha'(G) \geq \frac{1}{2}(n-k)$ iff $G \vee K_k$ has a perfect matching~~

Consider a set $A \subseteq V(G) \cup V(K_k)$ and consider

$$oc_{G'}(G'-A) - |A'|.$$

If $A' \not\supseteq V(K_k)$, then there is only one component. It is odd only if $|A|$ is odd.

(Since parity of k equals parity of n.)

Thus, if $oc(G'-A) > |A'|$, then $A' \supseteq V(K_k)$.

Let $A = A' \cap V(G)$, so $oc(G'-A') = oc(G-A) \leq |A| + k = |A'|$

Thus, Tutte's Condition holds on G', so G' has a perfect matching.

Hence G has a matching of size at least $\frac{1}{2}(n - \underbrace{k}_{\#\text{ things matched to } K_k})$

and $\operatorname{def}(G) \leq k$.

$\square$