Chapter 3: Matchings and Factors.

3.1. Matchings and covers.

Def: **Matching** (non-loops, no shared endpts)

Saturated

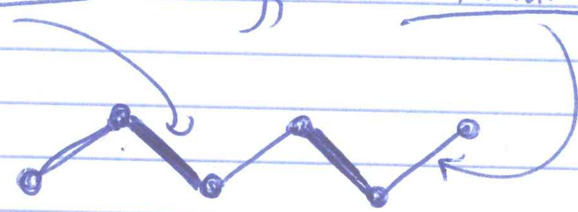
Perfect matching

Ex: Perfect matchings in $K_{n,n}$.

Perfect matchings in K_{2n}.

Def: **size**

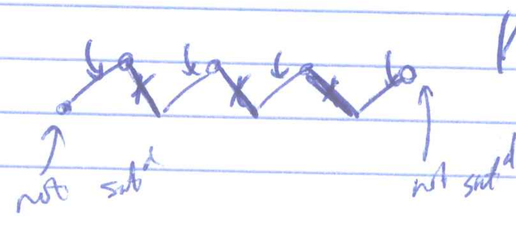
maximal matching, **maximum matching**.



Def: **M-alternating path** **M-augmenting path**

Prop: If M is maximum, then G has no M-augmenting path.

Pf: Let P be path. Let $M' = (M \cup E(P)) \setminus (E(P) \cap M)$.



$|M'| = |M| + 1$ and is a matching. $\square$

Def: <u>symmetric difference</u> $G \Delta H$

$M \Delta M'$ for matchings.

LMA: Every component of the symmetric difference of two matchings M, M' is a path or an even cycle.

Pf: Since M, M' are matchings, every vertex has ~~at most~~ one incident edge from either matching.

So, $\deg_{M \cup M'} v \leq 2$. So $M \cup M'$ is a collection of cycles and paths.

$M \Delta M'$ is the subgraph of $M \cup M'$ where the components isomorphic to K_2 are removed. $\square$

$|S| \leq |N(S)|$ (need enough qualified people to fill this set of jobs)

TONCAS.

Thm (Hall's Thm (1935)) An X,Y-bigraph G has a ~~perfe~~ matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Pf: (⇒) The ~~matched ed~~ edges in the matching saturating S must have their endpt in $N(S)$

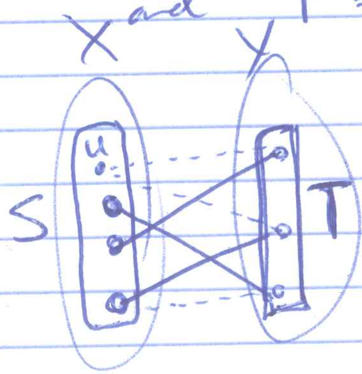
(⇐) Proof by contrapositive:

~~If $|N(S)| < |S|$ for some $S \subseteq X$ then G has no matching saturating X.~~

If M is a <u>maximum</u> matching in G and does not saturate X, then there exists $S \subseteq X$ w/ $|S| > |N(S)|$.

Let $u \in X$ be unsaturated by M.

Among vertices reachable by M-alternating path from u, let S be those vertices in X and T be those in X.



Observe $u \in S$.

Claim: M matches T w/ $S - u$.

The M-alternating paths reach Y by edges <u>not</u> in M and return to X by edges in X from a vertex in T!

If there is $v \in T$ unsaturated, then the M-alt path from u to v is <u>M-augmenting</u> and M is <u>not</u> maximum!

So, edges of M leaving T reach all elts of $S - u$.

By this matching, $T \subseteq N(S)$ AND $|T| = |S-u|$.

In fact, $T = N(S)$, since ~~otherwise the~~ all vertices in $N(S)$ are reachable by M-alternating paths:

(A) $v \in N(u)$, the edge $uv \notin M$ is M-alternating.

(B) $v \in N(x)$, let P be the M-alt path from u to x.
$x \in S-u$ P ends w/ an edge in M, so

$P + xv$ is also M-alt'g.

So, $T = N(S)$, $|T| = |S-u| = |S|-1$. $\blacksquare$

Rmk: Certificates for POSITIVE and NEGATIVE things!

When $|X| = |Y|$, Hall's Thm is the <u>Marriage Thm</u>
(Frobenius, 1917)

Cor: For $k>0$, Every k-regular X,Y-bigraph has a perfect matching.

Pf: Recall $|X| = |Y|$.

Let $S \subseteq X$ and count edges between S & $N(S)$ in two ways:

$$k|S| \leq k|N(S)|.$$

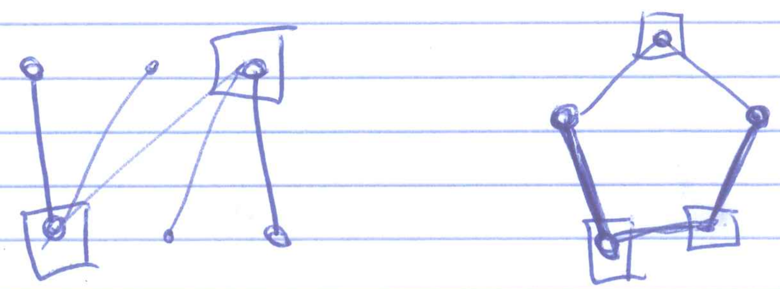
$$|S| \leq |N(S)|.$$

$\square$

Def: A <u>vertex cover</u> $Q \subseteq V(G)$ covers the <u>edges</u>.

Q <u>covers</u> $E(G)$.

Ex:



Thm (König 1931, Egerváry 1931): If G is bipartite graph, then the max size of a matching in G is the minimum size of a vertex cover in G.

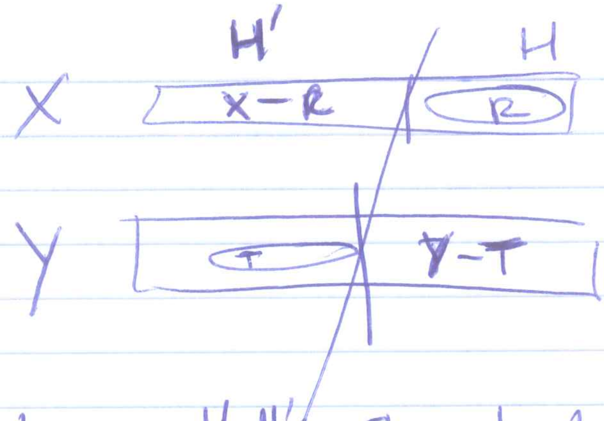
Pf: Let G be an X,Y-bigraph. Distinct vertices must be used for cover edges of a matching, so

$|Q| \geq |M|$ for all matchings $M \subseteq E(G)$ and vertex covers $Q \subseteq V(G)$.

Given a <u>smallest</u> vertex cover Q, we will construct a matching of size $|Q|$.

Partition Q by $R = Q \cap X$, $T = Q \cap Y$.

Let H, H' be the subgraphs of G induced by $R \cup (Y-T)$, $T \cup (X-R)$, respectively.



We use Hall's Thm to show that H has a matching that saturates R into $Y-T$ and H' has a matching that saturates T into $X-R$.

Since H & H' are disjoint, this makes a matching of size $|Q|$ in G.

Since $R \cup T$ is a vertex cover, ~~every edge~~ there are no edges from $X-R$ to $Y-T$.

For all $S \subseteq R$, consider $N_H(S) \subseteq Y-T$.

If $|N_H(S)| < |S|$, we can replace S by $N_H(S)$ into find a smaller vertex cover!

So, Hall's Condition holds.

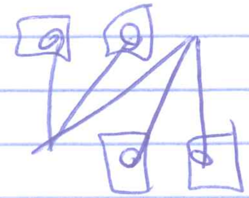
Similar for $S \subseteq T$ and H'. $\square$

Def: <u>min-max relation</u>

<u>dual optimization problems</u>

(What does equality mean?)

Def: independence #.



<u>edge covers</u> (dual)

Rmk: Isolated Vertices, P. Matching.

Def:

max indep set	$\alpha(G)$
max matching	$\alpha'(G)$
min vertex cover	$\beta(G)$
min edge cover	$\beta'(G)$

Rmk: König-Egerváry: $\alpha'(G) = \beta(G)$ for bipartite G.

$\alpha'(G) \le \beta(G)$.

$\alpha(G) \le \beta'(G)$.

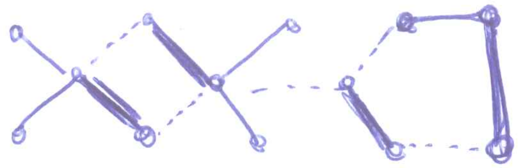
LMA: In a graph G, $S \subseteq V(G)$ is an independent set if and only if $\bar{S}$ $(= V(G) \setminus S)$ is a vertex cover.

~~Therefore,~~ And Hence $\alpha(G) + \beta(G) = n(G)$.

Pf: ($\Rightarrow$) If S indep, every edge is incident to something in $\bar{S}$.

($\Leftarrow$) If $\bar{S}$ covers edges, no edges lie in S. $\square$

S max $\Leftrightarrow$ $\bar{S}$ min

Ex: 

Thm (Gallai 1959): If G is a graph w/ no isolated vertices, then

$$\alpha'(G) + \beta'(G) = n(G).$$

Pf: From a maximum matching M, we construct an edge cover of size $n(G) - |M|$.
This will show $\beta'(G) \leq n(G) - \alpha'(G)$.

From a minimum edge cover L, we will construct a matching of size $n(G) - |L|$ showing $\alpha'(G) \geq n(G) - \beta'(G)$.

From M, add one edge for every unsaturated vertex.
This gives at most $n(G) - 2|M|$ new edges, so there are at most $n(G) - |M|$ edges.

From L, if both endpoints of an edge $e \in E(G)$ are covered by other edges in L, then $e \notin L$.

$\therefore$ Each component in the subgraph H w/ $E(H) = L$ has at most 1 vertex of degree higher than 1 (and is a star).

Let k be the # of these components.

$|L| = n(G) - k$ ← one edge for every leaf of a star.

Choose matching by selecting one edge from each star. □

Cor: If G is bipartite w/ no isolated vertices, then
$$\alpha(G) = \beta'(G).$$
□

$$\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$$

EQ by KE.

3.2 Algorithms and Applications

Now, we want to find a maximum matching!

Prop: A matching M is maximum in G iff there is no M-augmenting path.

Pf: (In Book).

Algorithm (Augmenting Path Algorithm):

Input: X,Y-bigraph G, matching M, set U of M-unsaturated vertices in X.

Idea: Explore M-alternating paths from U, with $S \subseteq X$, $T \subseteq Y$ "explored". As a vertex v is reached, let $p(v)$ be the preceding vertex.

Init: $S = U$, $T = \emptyset$.

Iteration: ① If S has no unmarked vertex, stop and report $T \cup (X - S)$ as a minimum cover, M as maximum matching.

② Otherwise, select unmarked vertex $x \in S$ for exploration:

for all $y \in N(x)$:

A) Consider $y \in N(x)$ w/ $xy \notin M$.

B) If y unsaturated, terminate and report an augmenting path from U to y (follow $p(x)$, $p(p(x))$, ... until U)

C) Otherwise, y is matched to some $w \in X$ by M.

D) Include y in T, w in S with $p(y) = x$, $p(w) = y$.

E) Repeat for all $y \in N(x)$ w/ $xy \notin M$.

(X to Y on $\overline{M}$, Y to X on M)

Thm: Iteratively applying the Augmenting Path Algorithm to a bipartite graph produces a matching and a vertex cover of equal size.

Pf: We only need to verify that the algorithm produces either:
① An M-augmenting path.
or
② A vertex cover of size $|M|$.

If ①, then we build a larger matching and repeat. Eventually, we will reach a maximum matching.

We must show that ② holds when the input is a max matching.

Claim: $Q = T \cup (X - S)$ is a vertex cover of size $|M|$.

To show it is a vertex cover, we need no edge from S to $Y - T$.

An M-alternating path from U enters X only on edges of M.

So, all vertices in $S - U$ are saturated to $\underline{T}$, so no edge of M matches S to $Y - T$.
Also, when $x \in S$ is visited, we explored all edges not in M and put those endpts in T.
So, no edges from S to $Y - T$, $\therefore Q$ is VC.

$U \subseteq S$, so $X - S$ is saturated and sat^d to $Y - T$!
T is sat^d to S, so

$|M| = |T| + |X - S| = |Q|$. $\square$

Def: Running time, Poly time

Big O notation? $O(nm)$ for aug. path

$O(\sqrt{n} \cdot m)$ possible.

Weighted Bipartite Matching

Place weights on the edges. (Can assume $K_{n,n}$ w/ appropriate 0's)

(Farms & Plants, Government Subsidies)

Def: traversal of an $n \times n$ matrix

Assignment Problem: max-sum traversal.

Same as weighted matching problem on $K_{n,n}$ if

$a_{ij} = w(x_i, y_j)$

A (weighted) cover consists of labels $u_1, \dots, u_n$ and $v_1, \dots, v_n$

such that $u_i + v_j \geq w(x_i, y_j)$ for all i,j.

We seek a minimum $\Sigma u_i + \Sigma v_j = c(u,v)$ (Cost of $\underline{u}, \underline{v}$)

LMA (Duality of weighted matching & covers):

For a perfect matching M and weighted cover $(\underline{u}, \underline{v})$,

~~in~~ in a weighted bipartite graph G, always

$$w(M) \leq c(u,v).$$

Also, $w(M) = c(u,v)$ if and only if M consists of edges where

$w_{i,j} = u_i + v_j$ (in this case they are optimal.)

Pf: Since M saturates every vertex, summing the constraints $w_{i,j} \le u_i + v_j$ for every edge $(x_i, y_j) \in M$ gives

$$w(M) \le c(\underline{u}, \underline{v}).$$

Also, $w(M) = c(\underline{u}, \underline{v})$, if and only if equality always holds. $\square$

Def: The equality subgraph $G_{u,v}$ for a weighted cover $(\underline{u}, \underline{v})$ is the spanning subgraph of $K_{n,n}$ whose edges are the pairs (x_i, y_j) where $u_i + v_j = w_{i,j}$.

In the cover, the excess is $\boxed{u_i + v_j - w_{i,j}}$

Alg (Hungarian Algorithm - Kuhn [1955], Munkres [1957]):

Input: Matrix of weights on edges of $K_{n,n}$ w/ bipartition X, Y.

Idea: Iteratively adjust cover (u,v) until the equality subgraph $G_{u,v}$ has a perfect matching.

~~Iteration~~: Init: Let (u,v) be a cover w/ $u_i = \max_j w_{i,j}$, $v_j = 0$.

Iteration: 1) Let M be maximum matching in $G_{u,v}$.

2) If M is perfect, output M, u, v.

3) Otherwise, let Q be a vertex cover of size $|M|$ in $G_{u,v}$. Let $R = Q \cap X$, $T = Q \cap Y$, and

$$\varepsilon = \min\{u_i + v_j - w_{i,j} : x_i \in X - R,\ y_j \in Y - T\}$$

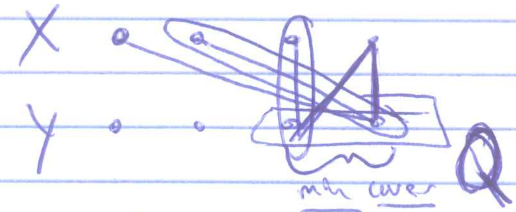
4) Decrease u_i by ε for $x_i \in X - R$, increase v_j by ε for $y_j \in Y - T$. (Repeat)

Instead, we compute w/ matrices.

$$
\begin{array}{c|cccc}
 & 0 & 0 & 0 & 0 \leftarrow v \\
\hline
4 & & & & \\
5 & & & & \\
3 & & & & \\
1 & & & & \\
\uparrow u & & & &
\end{array}
\quad
X \left\{ \begin{array}{c} 4 \\ 5 \\ 3 \\ 1 \end{array} \right.
\underbrace{\begin{pmatrix} 1 & 3 & 2 & 4 \\ 1 & 2 & 1 & 5 \\ 2 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}}_{Y}
\longrightarrow
X\text{-}R \left\{ \begin{array}{c} 4 \\ 5 \\ 3 \\ 1 \end{array} \right.
\begin{pmatrix} \overbrace{3 \quad 1}^{Y-T} & \overbrace{2 \quad 0}^{T} \\ 4 \quad 3 & 4 \quad \boxed{0} \\ 1 \quad 2 & \boxed{0} \quad 1 \\ 1 \quad 1 & 0 \quad 0 \end{pmatrix}
\begin{array}{l} \\ \text{Excess} \\ \leftarrow \text{matrix.} \\ \\ \end{array}
$$

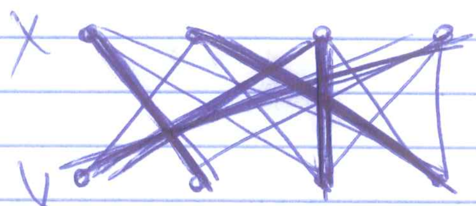
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T \quad T$$

max matching.



$\varepsilon = \min\{3, 1, 4, 3, 1, 2, 1, 1\} = 1$.

$$
\begin{array}{c|cccc}
 & 0 & 0 & 1 & 1 \\
\hline
3 & 2 & \boxed{0} & 0 & 0 \\
4 & 3 & 2 & 4 & \boxed{0} \\
2 & 0 & 1 & \boxed{0} & 1 \\
0 & \boxed{0} & 0 & 0 & 0
\end{array}
$$



perfect! We are done.

Thm: The Hungarian algorithm finds a max-weight matching and min-weight vertex cover.

Pf: The algorithm begins with a cover. It can terminate *only* when we get a perfect matching in the equality subgraph, which gives equality.

Suppose (u,v) is our current cover, and eq subgraph has no p.m. Then, ~~if~~ let (u',v') be the resulting cover from a cover Q.

Observe $\varepsilon > 0$ since it is the min of *positive* items.

First, we verify (u',v') is a cover.

Note: $u_i + v_j \geq w_{i,j}$ and

for $i \notin R = Q \cap X$, $\underbrace{u_i - \varepsilon}_{u_i'} + \underbrace{v_j}_{v_j'} \geq w_{i,j}$ iff $\underbrace{u_i + v_j - w_{i,j} \geq \varepsilon}_{\text{holds!}}$

Otherwise,

$$u_i' + v_j' = u_i + v_j \ (+\varepsilon?) \geq w_{i,j}.$$

Remains a cover!

If $w_{i,j}$ are all rational, multiply by least common denom to find an equivalent *integer* problem. Now $\varepsilon \geq 1$,

We *reduce* the cost of the cover by an integer amount.

In a finite # of steps, we will not be able to ~~extend~~ lower the cover, so there must be a terminating step.

$\square$