6.5 Total Unimodularity

Recall: $A^{-1} = \frac{1}{\det(A)} A^{*}$ where $A^{*}_{i,j} = \underline{\det(A_{-i,-j})}$

Prop: Let A be an integral, nonsingular $m \times m$ matrix.

Then $A^{-1}b$ is integral for every integral vector $\underline{b} \in \mathbb{R}^m$ iff $\det(A) = \pm 1$.

Pf: Suppose $\det(A) = \pm 1$, then A^{-1} is integral, so $A^{-1}\underline{b}$ is integral.

Conversely, let $A^{-1}b$ be integral for every integer vector $\underline{b}$.

Then in particular,

$$A^{-1}\underline{e}_1, \ A^{-1}\underline{e}_2, \ \dots, \ A^{-1}\underline{e}_m$$

are all integral, so A^{-1} is <u>integral</u>.

Thus, $\det(A)$ & $\det(A^{-1})$ are <u>both</u> integral.

Since $\det(A) \cdot \det(A^{-1}) = 1$, $\det(A) = \det(A^{-1}) \in \{+1, -1\}$. $\blacksquare$

Def: A (an $m \times n$ mtx) is <u>unimodular</u> if every $m \times m$ submatrix B has $\det(B) \in \{+1, -1\}$.

Thm: Let A be an integral $m \times n$ mtx of full row rank. Then every basic feasible solution to the LP

$$A\underline{x} = \underline{b}$$
$$\underline{x} \geq 0$$

is integral for every integral vector $\underline{b} \in \mathbb{R}^m$

iff A is unimodular.

Pf: Suppose A is unimodular and $b \in \mathbb{R}^m$ is integral.

A bfs x^* is given by the basis mtx B, an $m \times m$ submtx of A. Thus, $\det(B) \in \{+1, -1\}$, and

$$B x^*_B = b \iff x^*_B = \underbrace{B^{-1} b}_{\text{integral by Prop.}}$$

$x^*_i = 0$ for i not in basis.

Now, consider any $m \times m$ submtx B, and any vector $v \in \mathbb{R}^m$, integral. To show $\det(B) \in \{+1, -1\}$, it suffices to show that $B^{-1}v$ is integral (by Prop). So, let y be integral s.t.

$$y + B^{-1} v \geq 0 \text{ and let } b = B(y + B^{-1}v) = \underline{By + v}$$

Observe that $\underline{b}$ is integral. Also, the solution to

$$A\underline{z} = \underline{b}, \ \underline{z} \geq \underline{0} \quad \text{exists, w/ } \underline{z_B = y + B^{-1}v}.$$

Since z_B is integral,

$y - B^{-1}v$ is integral,

but y was integral, so $B^{-1}v$ is integral. $\square$

Def: A, $m \times n$ mtx, is <u>totally unimodular</u> (TUM)

if <u>every</u> square submtx has determinant in $\{0, +1, -1\}$.

(In particular, all <u>entries</u> of A are $0, +1, -1$.)

Moreover: A is TUM $\Leftrightarrow$ $[A \mid I_m]$ is UM. (Exercise!)

Thm: (Hoffman-Kruskal Thm): Let A be ~~an~~ $m \times n$ integral matrix. Then all bfs's to $A\underline{x} \leq \underline{b}$, $\underline{x} \geq \underline{0}$ are integral iff A is TUM.

Pf: Since bfs's to $A\underline{x} \leq \underline{b}$, $\underline{x} \geq 0$ correspond to bfs's of $[A \mid I]\binom{\underline{x}}{\underline{s}} = \underline{b}$, $\underline{x}, \underline{s} \geq 0$,

the result follows from previous thm. $\square$

~~Thm: Let A b~~

Q: How can we tell if A is TUM?

Thm 6.27: Let A be a $0, \pm 1$ - valued $m \times n$ mtx where each column has at most one $+1$ and at most one -1. Then A is TUM.

Pf: By induction on $1 \leq k \leq m$.

$k=1$: All entries are $\{0, +1, -1\}$, so $\det(B) \in \{0, +1, -1\}$ $\forall$ 1×1 mtx's.

$k \geq 2$: Let B be a $k \times k$ submtx.

If B has a column with at most one nonzero, then expanding on B in that j^{th} column shows

$$\det(B) = \sum_{i=1}^{k} \underbrace{b_{i,j}} \cdot \det(B_{-i,-j}) = \underbrace{b_{i,j}}_{\in \{0, +1, -1\}} \cdot \underbrace{\det(B_{-i,-j})}_{\in \{0, +1, -1\}} \in \{0, +1, -1\}.$$

Otherwise: Every column of B has exactly one $+1$ and exactly one -1.

Thus, the row sum is 0. & $\det(B) = 0$. $\square$

Total Unimodularity

Why integer solutions to LPs?

Def: Square, integer matrix B is unimodular (UM) if $\det B \in \{+1,-1\}$.

An integer matrix A is totally unimodular (TUM) if every square, nonsingular submatrix of A is UM.

$\hookrightarrow$ i.e. if B is square submatrix, then $\det B \in \{-1,0,+1\}$.

Let A be TUM $m \times n$ mtx and B a ~~sub~~ non-singular square matrix formed by m lin. indep cols of A.

Defines $\underline{x} = B^{-1}\underline{b} = \dfrac{B^{adj}\,\underline{b}}{\det B}$.

Since B^{adj} is integer, and $\det B \in \{+1,-1\}$, then if $\underline{b}$ is integer then $\underline{x}$ is integer.

Def: $R_1(A,b) = \{\underline{x} : A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0}\}$

(the standard-form polytope)

Thm 13.1: If A is TUM, then all vertices of $R_1(A,b)$ are integer for any integer vector $\underline{b}$.

We can also use inequality constraints:

$$R_2(A,b) = \{\underline{x} : A\underline{x} \leq \underline{b},\ \underline{x} \geq \underline{0}\}$$

Thm 13.2: If A is TUM, then all vertices of $R_2(A,b)$ are integer for any integer vector $\underline{b}$.

Pf: If A is TUM, then so is $\underline{(A \mid I_m)}$

and $R_2(A,b) \longleftrightarrow R_1((A \mid I_m), \underline{b})$

Let C be square, nonsingular submatrix of $(A \mid I)$.

The $\underline{\text{rows}}$ can be permuted so we have

$$\ell \left\{ \underbrace{\begin{pmatrix} B & O \\ \hline D & I_k \end{pmatrix}}_{\ell} \right. \qquad \text{for some } k \le \ell.$$

$$\det C = \det B \cdot \det \underline{I_k} = \det B \in \{+1, -1\}$$

$\square$

Thm 13.3 An integer mtx A w/ $a_{ij} \in \{+1, 0, -1\}$ is TUM if no more than two nonzero entries appear in any column, and if the rows can be partitioned into two sets I_1 & I_2 s.t.:

1. If a column has two entries of the same sign, their rows are in different sets, and

2. If a column has two entries of different signs, their rows are in the $\underline{\text{same}}$ set.

Pf: By induction on size of submatrices.

Since $a_{ij} \in \{+1, 0, -1\}$, every 1×1 submatrix is UM or singular.

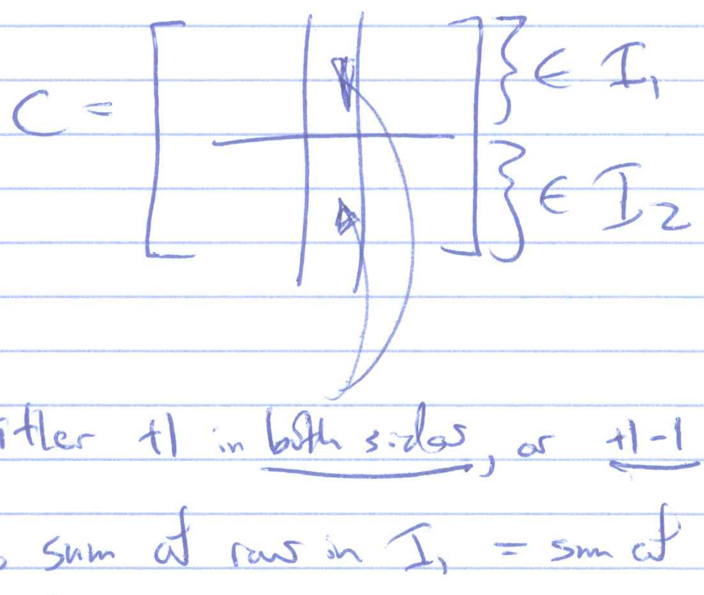
Let C be $k \times k$. If any column has all zeros, then singular.

If C has a column w/ 1 nonzero entry, we Cramer's Rule:

$$C = \left[\begin{array}{c|c} \pm 1 & xxxx \\ \hline 0 & C' \end{array}\right] \qquad \det C = \pm \det C' \in \{+1, 0, -1\}.$$

Now, suppose C has 2 entries in every column.

Split by I_1 & I_2



Either $+1$ in both sides, or $+1$ -1 & 0,

So, sum of rows in I_1 = sum of rows in I_2.

Hence, $\det C = 0$. $\square$

Cor: Any LP in standard or canonical form whose constraint matrix A is either

1. The node-arc incidence matrix of a directed graph, or
2. The node-edge incidence matrix of an undirected bipartite graph,

has <u>only</u> integer optimal vertices.

This includes:

- Shortest Paths
- Max-flow → Some work to be done!
- Weighted bipartite matching

Pf: 1. Use $I_1 =$ all rows, $I_2 = \emptyset$.

2. Use $I_1 = X$, $I_2 = Y$. $\square$

Chapter 13: Integer Programming.

Standard Form:

$$\min \ \underline{c}^T \underline{x}$$

$$\text{s.t.} \quad A\underline{x} = \underline{b}$$

$$\underline{x} \geq \underline{0}$$

$$\underline{x} \text{ integer}$$

We assume entries in A & $\underline{b}$ are integer (Rational is good enough.)

Why Integer? Why Not Round?

Ex: