# Tests for convergence/divergence of series: §§10.2-10.6

| Name | Form | Converges if and only if | Total Sum |
|---|---|---|---|
| Geometric | $\sum_{n=1}^{\infty} ar^{n-1}$ | $\lvert r\rvert < 1$ | $\frac{a}{1-r}$ |
| $p$-series | $\sum_{n=1}^{\infty} \frac{1}{n^p}$ | $p > 1$ | – |

**$n$-th Term Test:** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_n a_n$ diverges.

**Alternating Series Test:** If $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, then $\sum_n (-1)^n b_n$ converges.

**(Absolute) Ratio Test:** Let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, then

- If $\rho < 1$, then $\sum_n a_n$ converges.
- If $\rho > 1$, then $\sum_n a_n$ diverges.
- If $\rho = 1$, then the test is inconclusive.

**NOTE:** Ratio Test is good if $a_n$ contains a factorial or an exponential.

**Limit Comparison Test (LCT):** Given positive series $\sum_n a_n$ and $\sum_n b_n$, let $L = \lim_{n\to\infty} a_n/b_n$.

- If $0 < L < \infty$, then $\sum_n a_n$ and $\sum_n b_n$ converge or diverge together.
- If $L = 0$ and $\sum_n b_n$ converges, then $\sum_n a_n$ converges.
- If $L = \infty$ and $\sum_n b_n$ diverges, then $\sum_n a_n$ diverges.

**NOTE:** LCT is good for rational functions.

**Integral Test:** Let $a_n = f(n)$ where $f(n)$ is (1) positive, (2) decreasing and (3) continuous on $[N, \infty)$. Then $\sum_{n\geq N} a_n$ and $\int_N^\infty f(x)\,dx$ converge or diverge together.

**Telescoping/Collapsing Series:** Use partial fractions to allow terms in the series to cancel and get a closed form for $S_n$. Check directly if $\lim_{n\to\infty} S_n$ exists.

**NOTE:** Sums can be computed only for **geometric** series, **telescoping** series and those derived from Taylor series (§9.8).

**Ordinary Comp. Test (OCT):** Let $0 \le a_n \le b_n$, for $n \ge N$.

- If $\sum_n b_n$ converges, then $\sum_n a_n$ converges.
- If $\sum_n a_n$ diverges, then $\sum_n a_n$ diverges.

**NOTE:** LCT beats OCT unless you have a sin, cos or arctan function.

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# Estimates

**Integral Test:** Let $a_n = f(n)$ where $f(n)$ is (1) positive, (2) decreasing and (3) continuous on $[N, \infty)$.

$$\int_{n+1}^{\infty} f(x)\,dx \le S - S_n = \sum_{k=n+1}^{\infty} a_k \le \int_{n}^{\infty} f(x)\,dx$$

**Alternating Series Test:** If $b_{n+1} \le b_n$ and $\lim_{n\to\infty} b_n = 0$, then $|S - S_n| = \left| \sum_{k=n+1}^{\infty} (-1)^n b_n \right| \le b_{n+1}$.

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# Sequences

**Squeeze Theorem:** If $a_n \le b_n \le c_n$ and $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then $L = \lim_{n\to\infty} b_n$.

**Monotonic Sequence:** If $\{a_n\}$ is nonincreasing and bounded below, it converges.
If $\{c_n\}$ is nondecreasing and bounded above, it converges.