MATH413 MIDTERM 2

March 16 10:00-10:50am **Name: ....**

Answer as many problems as you can. Show your work. An answer with no explanation will receive no credit. Write your name on the top right corner of each page.

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6

1: Determine the number of permutations of $[n] = \{1, \dots, n\}$, in which exactly three integers are in their own positions.

Solution:

Pick the three that are fixed and all other are not on their positions. So the number of permutations is

$$\binom{n}{3} D_{n-3},$$

where D_k is the number of permutations on k elements without fixed point - derangements.

Name: ....

2: Let n be a positive integer. Prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m\binom{2m}{m} & \text{if } n = 2m. \end{cases}$$

Hint: consider $(1-x^2)^n = (1+x)^n(1-x)^n$

Solution:

Suppose $n = 2m$. Consider coefficient of x^n of $(1-x^2)^n = (1+x)^n(1-x)^n$. It is

$$(-1)^m\binom{2m}{m}x^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k x^k \cdot \binom{n}{n-k}x^{n-k}$$

$$(-1)^m\binom{2m}{m}x^n = \sum_{k=0}^{n}\binom{n}{k}^2(-1)^k x^n$$

$$(-1)^m\binom{2m}{m} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^2$$

Now consider that n is odd. Observe that for every k

$$(-1)^k\binom{n}{k}^2+(-1)^{n-k}\binom{n}{n-k}^2 = ((-1)^k+(-1)^{n-k})\binom{n}{k}^2 = (1-1)\binom{n}{k}^2 = 0$$

because n is odd and hence k and $n-k$ have different parity.

Name: ....

3: Determine the number of integer solutions of $x_1 + x_2 + x_3 \leq 40$ where $1 \leq x_1, x_2, x_3 \leq 20$.
(If you struggle too much with ≤ 40, you may solve $= 40$ for partial credits.)

Solution:
Rewrite as

$$y_1 + y_2 + y_3 + y_4 = 37$$

where

$$0 \leq y_1, y_2, y_3 \leq 19 \text{ and } 0 \leq y_4$$

Notice that it cannot happen that $y_i \geq 20$ and $y_j \geq 20$ at the same time. Hence the number of solutions is using principle of inclusion and exclusion

$$\binom{37+3}{3} - 3 \cdot \binom{17+3}{3}$$

because $\binom{17+3}{3}$ is the number of solutions if condition $y_i \geq 20$.

Name: ....

4: A host wants to seat n couples in a table, seating the men first in a specified order on every second seat. However, the host does not want to put wives on either side of their husband. How many ways are there to do this? *Clarification: So men and women are alternating in the seating. Men are already seated.*

Solution:

We convert the problem to a rook placement problem with forbidden board:

$\times$	$\times$				
	$\times$	$\times$			
		$\times$	$\times$		
			$\times$	$\times$	
				$\times$	$\times$
$\times$					$\times$

Let A_s be the set of placements where a rook stands on forbidden square s for each forbidden square s. Hence we have $2n$ such sets. The solution is

$$\left|\overline{A_1} \cap \overline{A_1} \cap \cdots \cap \overline{A_{2n}}\right| .$$

Counting intersections is a bit troublesome but not impossible. We count intersections of size k all at the same time. Let $r_k(n)$ be the number of ways to place k rooks in the forbidden region of $n \times n$ board. So it corresponds to picking k out of $2n$ positions. If you number them as in the figure below it means pick k of $2n$ such that no two are consecutive (including cyclic order).

1	2				
	3	4			
		5	6		
			7	8	
				9	10
12					11

First suppose that position $2n$ ($= 12$ in example) is not picked. Then we can compute the number by picking k from $2n-k$ and adding a not picked position after every picked. The number of such picking is $\binom{2n-k}{k}$. Now suppose that $2n$ is picked. Then we need to remove the first square in order to avoid cyclic conflict and pick $k-1$ from the rest. So we have $\binom{2n-2-(k-1)}{k-1} = \binom{2n-k-1}{k-1}$. By summing we get

$$r_k(n) = \binom{2n-k}{k} + \binom{2n-2-(k-1)}{k-1} = \frac{2n}{2n-k}\binom{2n-k}{k}.$$

Now we can use inclusion and exclusion and show that

$$\left|\overline{A_1} \cap \overline{A_1} \cap \cdots \cap \overline{A_{2n}}\right| = \sum_{k=0}^{n} (-1)^k r_k(n)(n-k)!$$

and hence

$$\sum_{k=0}^{n} (-1)^k r_k(n)(n-k)! = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!$$

Note $(n-k)!$ comes from the number of possible extensions of placing k rooks to forbidden region to placing all n rooks.

Note that the question was almost *problème des ménages.*

Name: ....

5: There are 30 video game players. 15 of them play Legend of Zelda, 17 of them play Call of Duty and 20 of them play World of Warcraft. Legend of Zelda and Call of Duty is played by 8 players, Call of Duty and World of Warcraft is played by 10 players and World of Warcraft and Legend of Zelda also by 10 players. How many players play all three games?

Solution:
By principle of inclusion and exclusion we have

$$30 = 15 + 17 + 20 - 8 - 10 - 10 + x$$

where x is the number of players playing all three games. Hence the result is that all three games are played by 6 players.

Name: ....

6: Let n and k be a positive integers. Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Solution:

Suppose we have two sets A and B where $|A| = |B| = n$. Then the right hand side can be interpreted as *Choose one element x from A and from the rest $(A \cup B) \setminus \{x\}$ pick $n-1$ elements.* So the result is an n element subset of $A \cup B$ with one marked element from A.

We can view the right hand side in the same way but partitioned according to the size of subset taken from A. So $k\binom{n}{k}^2 = k\binom{n}{k}k\binom{n}{n-k}$ is the number of possibilities to pick an n element subset S of $A \cup B$ such that $|S \cap A| = k$. If we sum over all k, we get the total number of possibilities. Note that the sum starts at $k = 1$ so there is always a choice for the special element of A.

Name: ....

7: What is the number of ways to place four nonattacking rooks on the 4-by-4 boards without forbidden positions are marked by $\times$?

		$\times$	
$\times$	$\times$	$\times$	
	$\times$		$\times$

Solution:
Let A_i be the set of rook placements where a rook is placed in a forbidden position in row $1 \le i \le 3$ (where the rows are numbered from top to bottom, as usual). We need to compute $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$ by the complementary form of the inclusion-exclusion principle.

We have

$$|A_1| = 1 \cdot 3!, |A_2| = 3 \cdot 3!, |A_3| = 2 \cdot 3!$$

$$|A_1 \cap A_2| = 1 \cdot 2 \cdot 2!, |A_1 \cap A_3| = 1 \cdot 2 \cdot 2!, |A_2 \cap A_3| = (2 \cdot 2 + 1)2!$$

and

$$|A_1 \cap A_2 \cap A_3| = (1 \cdot 1 \cdot 2 + 1 \cdot 1 \cdot 1)1!.$$

Hence the number of placements is

$$4! - 6 \cdot 3! + 18 - 3 = 24 - 36 + 18 - 3 = 3.$$

Paper for attempts.