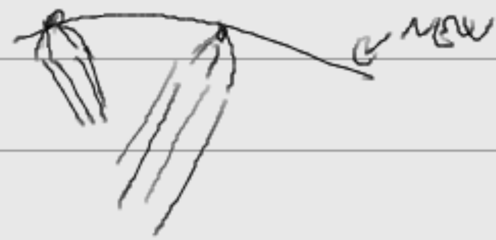
PROJECTIVE PLANES

PLANE + POINTS IN INFINITY FOR 2D



DEF:

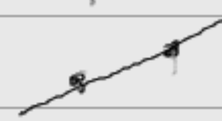
PROJECTIVE PLANE $P = (X, L)$

X ... POINTS

L ... $\subseteq P(X) = 2^X$ ... LINES

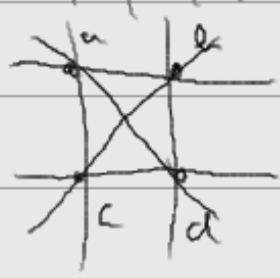
(P1) $\forall p \neq q \in L : |p \cap q| = 1$

(P2) $\forall x \neq y \in X : \exists! \, p \in P \quad x, y \in p$

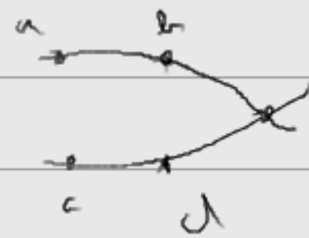


(P3) $\exists$ 4 POINTS IN GENERAL POSITION

$\exists a, b, c, d \in X$ ST NO TRIPLE ON 1 LINE



JUST



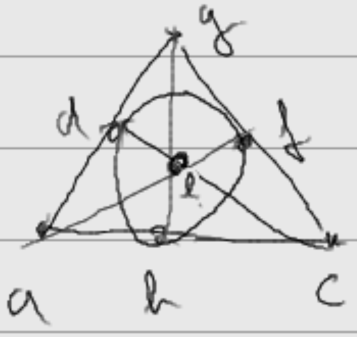
(P2') $\forall x \neq y \; \exists p \in P : x, y \in P$

(P2) => (P2')

(P2') + (P1) => (P2)

NOTE.. LINES ARE NOT REALLY LINES IN 2D
(NOT STRAIGHT)

EX FANO PLANE



$X = \{a, b, c, d, e, f, g\}$

$L = \{ \ldots$

$$I = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \left(\begin{array}{c} \cdots \\ \\ \\ \\ \\ \\ \cdots \end{array} \right.$$

BIBD

b .. 7 BLOCKS

v .. 7 VARIETIES

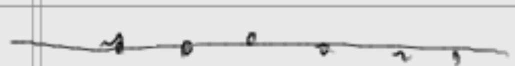
k .. 3 IN BLOCK

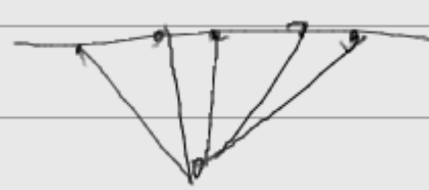
r .. 3 IN VAR IN BL.

λ .. 1 IN COMMON

↑ SPECIAL BIBDS

PATOLOGIC CASES - NOT (P3)





THM: $\forall$ PROJECTIVE PLANE (X, L)

EXISTS n SUCH THAT

1, $\forall \ell \in L: |p| = n+1$

2, $\forall x \in X: |\{p : x \in p \in L\}| = n+1$

3, $|X| = |P| = n^2 + n + 1$

n CALLED ORDER OF P

BIBD $b = v = n^2 + n + 1$

$k = r = n+1$

$\lambda = 1$

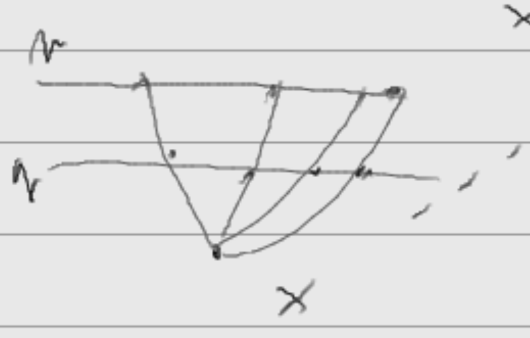
$n = 2$ IN FANO PLANE

PROOF

1, 2, AS MADE ON EXAM :-)

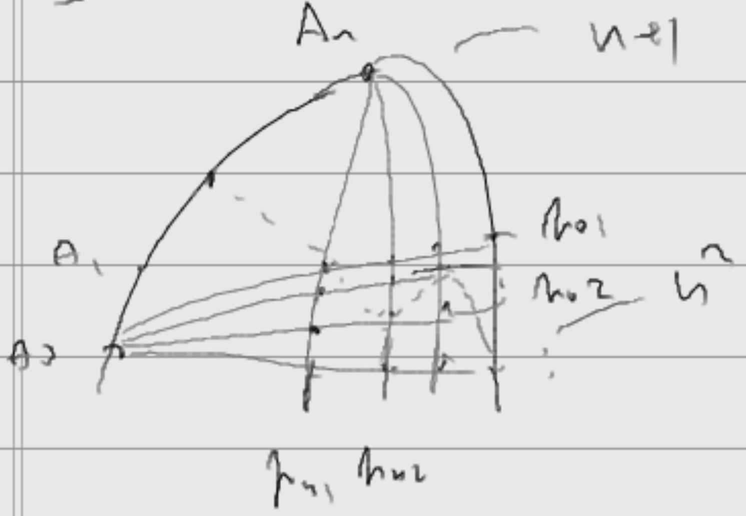
LINES p, q AND $x \notin p, q$

$x \in X$



COUNTING

3, LINE OF SIZE $n+1$



EVERY OTHER IN THESE n^2 POINTS

TOTAL .. $n+1+n^2$ POINTS

$1+n(n+1) = n^2+n+1$ LINES

DEF DUAL OF PROJECTIVE PLANE

$P(X, L)$

P' IS $(L, \{p_x \mid x \in X\})$

$\underline{I}_{P'} = \underline{I}_P^T$ ... INCIDENCE MATRIX

THM:

DUAL OF PROJECTIVE PLANE IS AGAIN A PROJECTIVE PLANE

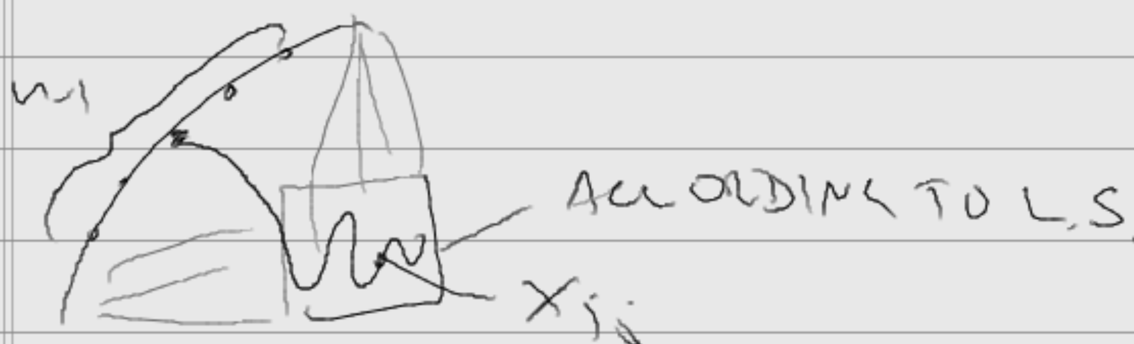
EXERCISE .. VERIFY AXIOMS $1 \Leftrightarrow 2$, $3 \Leftrightarrow 3$

THM:

PROJECTIVE PLANE OF ORDER n EXISTS IFF THERE ARE $n-1$ MOLS OF ORDER n.

PROOF:

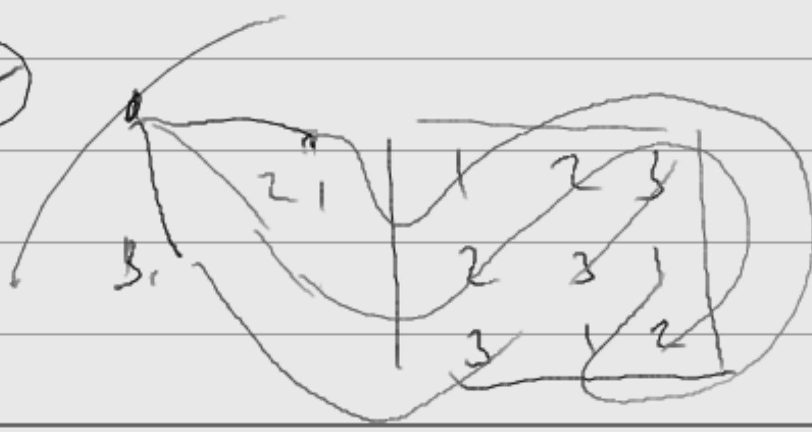
IDEA:



$$L^1 L^2 \dots L^{n-1} \in \{1, 2, \dots n\}^{n \times n}$$

$$L^k_{ij} = l \iff p_{kl} \ni X_{ij}$$

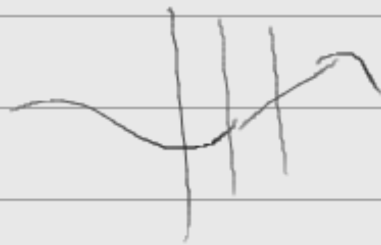
EX



PROJ. PLANES : L^k IS LATIN SQUARE

IN LINES DISTINCT ON ROWS

COLUMN



ONCE EACH.

ONE EACH

By L^k & L^m ARE ORTHOGONAL $(k \neq m)$

$\forall (l, h)\ \exists i, j$ S.T. $L^k_{ij} = l$ & $L^m_{ij} = h$

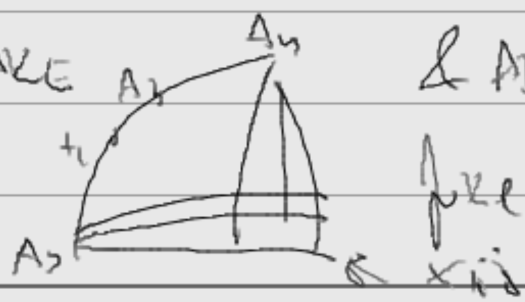
ENTRIES IN L's

PICK FROM A_k p_{kl} AND FROM A_m p_{mh}

$\Rightarrow p_{kl} \cap p_{mh} = (i,j)$ - IN $\square$

n-1 MOLS $\Rightarrow$ PROJ. PLANE

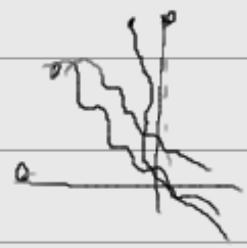
MAKE



& ADD

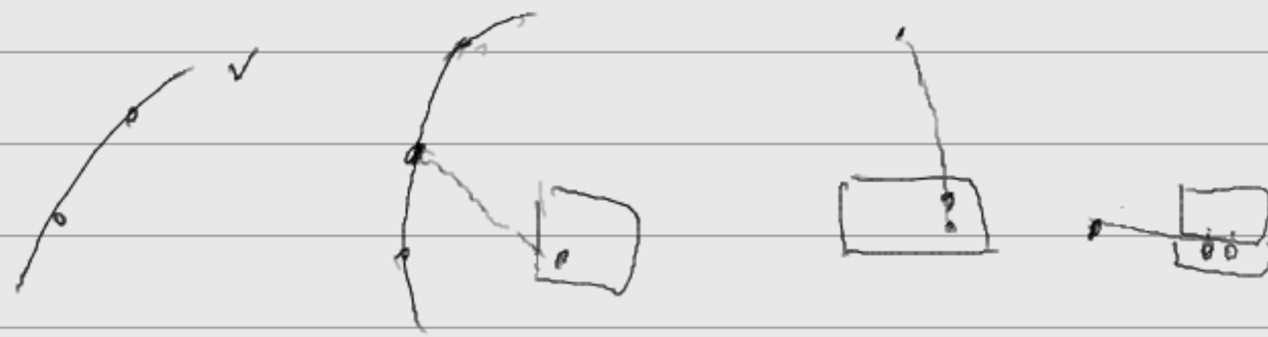
$p_{kl} = \{A_k\} \cup \{x_{ij} \mid L^k_{ij} = l\}$

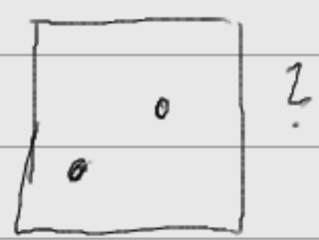
1) $\forall p,q : |p \cap q| = 1$ $\quad p,q \in$ LINES



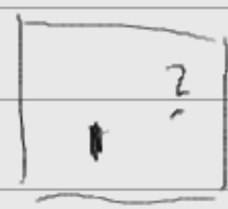
6 TYPES OF INTERSECTION

2) $\forall x,y \in X \ \exists p : xy \subseteq p$

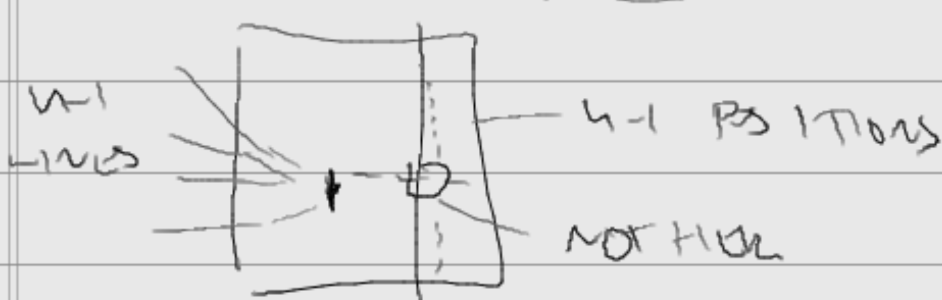




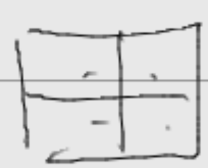
PERMUTE SQUARES



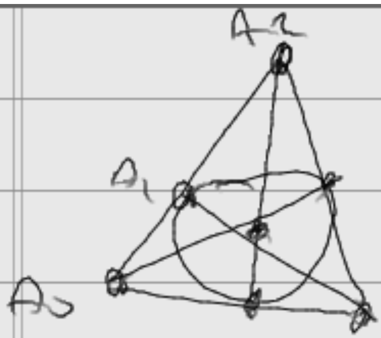
$\rightarrow$? ALWAYS DIFFERENT

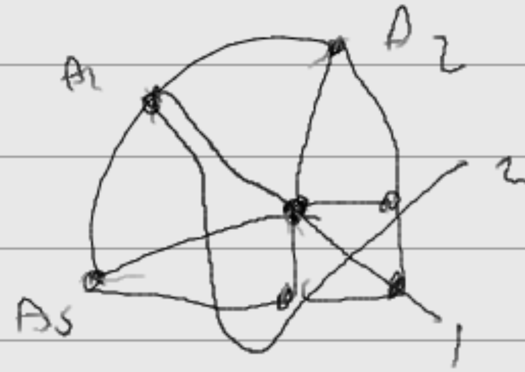


3)



$\leftarrow$ THESE 4 POINTS





$\begin{array}{cc} 1 & 3 \\ 2 & 1 \end{array}$

CONSTRUCTION FROM FINITE FIELDS:

THM:

LET $n = p^R$ WHERE IS PRIME
THEN EXISTS A PROJECTIVE PLANE
OF ORDER n.

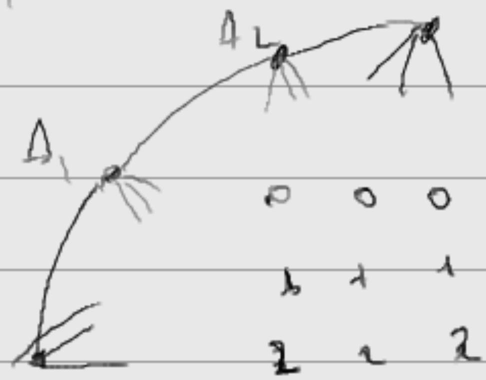
PROOF:

USE FINITE FIELD $F(n) = \{1, 2, .. n\}$
TABLES FOR ⊕ AND ⊗ .. FEW LECTURES BACK

$m_{kl} = \{A_k\} \cup \{X_{kj+l,j} \mid j = 1..n\}$

GRAPH OF A FUNCTION
$kj + l$

$\boxed{\text{EX}}$ $n = 3$



$M_{kl} = \{A_k\} \cup \{X_{kj+l,\,j} \mid j = 1,2,3\}$

$k=1$

$M_{10} = 1, 2, 0$

$M_{11} = 2, 0, 1$

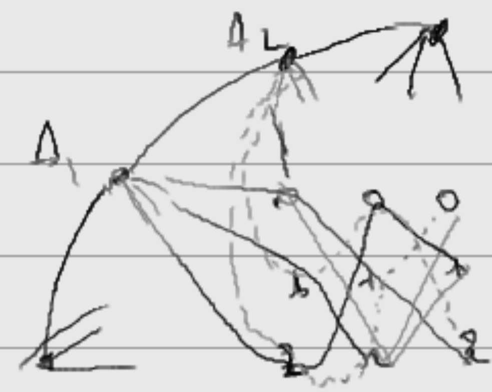
$M_{12} = 0\ 1\ 2$

$A_2 : \ k = 2$

$M_{20} : \ 2\ 1\ 0$

$M_{21} : \ 0\ 2\ 1$

$M_{22} : \ 1\ 0\ 2$



✗ BOUND 2 HOLES

AXIOM VERIFICATION:

(P1)

$$M_{kl} \cap M_{yz}$$

$$X_{kj+l,\,j} = X_{yj+z,\,j} \qquad \checkmark \text{ SOLVE } j$$

$$kj + l = yj + z$$

HAS SOLUTION → $j = (z - l)\cdot(k - y)^{-1}$

(P2)

$X_{ij} \quad X_{ij'}$

$\exists\ p_{kl}$: $X_{kj+l\,ij}$ TAKING BOTH

$$kj + l = i$$
$$kj' + l = i'$$
$$k(j - j') = i - i'$$
$$k = (i - i')\cdot(j - j')^{-1}$$

l IS EASY TO FINISH

$\square$

NOTES:

PROJECTIVE PLANE OF ORDER

2	YES (FANPLANO)	8	YES	2^3
3	YES .. WE MADE IT	9	YES	3^2
4	YES .. YES F(4)	10	$\boxed{\text{NO}}$	2xMOLS COMP
5	YES .. $\mathbb{Z}_5$	11	YES	
6	NO	12	???	
7	YES		2x⊥ MOLS	