

Math-484 List of definitions and theorems

This is a list of definitions that a student of 484 is required to know.

Definitions (Midterm 1): (*Define for a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ or a subset of \mathbb{R}^n*)

local minimizer, global minimizer, strict local minimizer, strict global minimizer, local maximizer, global maximizer, strict global maximizer, strict global maximizer, critical point, distance of two vectors, ball, interior point, for set D what is D^0 , open set, closed set, bounded set, compact set, gradient, Hessian, quadratic form, positive semidefinite matrix, positive definite matrix, negative semidefinite matrix, negative definite matrix, indefinite matrix, k^{th} principal minor, saddle point, coercive function, eigenvalue of a matrix, eigenvector of a matrix, convex set, closed half space, open half space, convex polyhedron, convex combination, convex polytope, convex hull, convex function, strictly convex function, concave function, strictly concave function, posynomial, unconstrained geometric program, dual of unconstrained geometric program, feasible solution, consistent program

Definitions (Midterm 2):

- best least squares k th degree polynomial *page 135*
- linear regression line *page 135*
- best least squares solution of (inconsistent) linear system *page 136*
- generalized inverse of a matrix A *page 136*
- inconsistent linear system *page 136–137*
- orthonormal vectors *page 138*
- subspace of \mathbb{R}^n *page 141*
- orthogonal complement of a subspace of \mathbb{R}^n *page 142*
- P_M - orthogonal projection of \mathbb{R}^m onto M *page 144*
- underdetermined system of linear equations *page 145*
- H -inner product *page 149*
- H -norm *page 149*
- H -orthogonal vectors *page 149*
- H -orthogonal complement *page 149*
- H -generalized inverse *page 150*
- hyperplane H in \mathbb{R}^n *page 158*
- boundary point of $C \subset \mathbb{R}^n$ *page 158*
- closure \bar{A} of $A \subset \mathbb{R}^n$ *page 163*
- subgradient of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ *page 168*
- subdifferential of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ *page 168*
- feasible vector of a program (P) *page 169*
- feasible region of a program (P) *page 169*
- consistent program (P) *page 169*
- superconsistent program (P) *page 169*
- convex program and dual convex program *pages 169, 200, 201*
- supremum of a real valued function defined on $C \subseteq \mathbb{R}^n$ *page 170*

- infimum of a real valued function defined on $C \subseteq \mathbb{R}^n$ page 170
- MP for program (P) - also define (P) page 171
- $MP(z)$ for program $(P(z))$ - also define $(P(z))$ page 171
- sensitivity vector of a program (P) page 177
- Lagrangian $L(\mathbf{x}, \lambda)$ of a program (P) - also define (P) page 182
- complementary slackness conditions - also define (P) page 184
- constrained geometric program (GP) and its dual (DGP) page 193

Definitions (Midterm 3):

- dual of a convex program page 200
- duality gap page 209
- absolute value penalty function page 217
- Courant-Beltrami penalty function page 219
- generalized penalty function page 223
- Jacobian Matrix of a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ page 85
- describe Newton's method for function minimization page 88, 3.1.3
- Descent method 3.2.6
- describe Steepest descent method page 98, 3.2.1
- outer product or tensor product page 115
- describe Broyden's method page 117, 3.4.1
- distance between two matrices page 118, 3.4.3
- describe BFGS method page 125, 3.5.3
- describe DFP method page 127, 3.5.4

Definition (SDP) and Interior Point Method

- Trace of a matrix A
- dot product for two matrices A and B
- A general form of (SDP)
- Dual semidefinite program $(DSDP)$
- Strictly feasible (SDP) and $(DSDP)$
- Write a convex program (P) and a barrier function corresponding to it.
- Give deformation of KKT conditions that holds for the barrier function in interior point method

Theorems and statements (Midterm 1): (*Try to do not ignore assumptions - like that sometimes the function must be continuous etc.*)

- state State Cauchy-Swartz inequality
- What can you tell about minimizers and maximizers of continuous $f : I \rightarrow \mathbb{R}$ where $I \subset \mathbb{R}$ is a closed interval? (Theorem 1.1.4 with proof)
- Which minimizers or maximizers of $f : D \rightarrow \mathbb{R}$ must be critical points? (I ask for Theorem 1.2.3?)
- State the cornerstone theorem for using gradient and Hessian for finding minimizers. (I

ask for Theorem 1.2.4)

- How can H_f help identify global minimizers and maximizers? (I ask for Theorem 1.2.5? or 1.2.9)
- What is the relation between convexity of f and Hf ? (I ask for Theorem 22)
- How principal minors of matrix A correspond to positive(negative) (semi)definiteness or indefiniteness of A ? (I ask for Theorem 1.3.3)
- How can H_f help identify local minimizers and maximizers? (I ask for Theorem 1.3.6, with proof)
- Do coercive functions have some special properties related to minimizers? (I ask for Theorem 1.4.4, with proof)
- Do coercive functions have some special properties related to maximizers?
- Is there any connection between $co(D)$ and set of all convex combinations of vectors from D ? ($D \subseteq \mathbb{R}^n$)
- Do coercive functions have some special properties related to minimizers?
- How eigenvalues of a symmetric matrix A correspond to positive/negative (semi)definiteness of A ?
- How principal minors of a symmetric matrix A correspond to positive/negative (semi)definiteness of A ?
- What is relation of convex hull of D and the set of all convex combination of vectors from D ? (Theorem 2.1.4)
- Is there a convex function $\mathbb{R} \rightarrow \mathbb{R}$ that is not continuous? (Theorem 2.3.1)
- Do (local) minimizers of a convex function have some nice properties? (Theorem 2.3.4 with proof)
- Do (local) maximizers of a concave function have some nice properties? (Theorem 2.3.4)
- Can a convex function be recognized by its gradient? (Theorem 2.3.5)
- Do critical points of convex functions have some nice properties? (Theorem 2.3.5 + Corollary)
- What is the correspondence between Hessian and convexity of a function? (in \mathbb{R}^n) (Theorem 2.3.7)
- Is it possible to decide if a function is convex by decomposing it to simpler ones? How? (Theorem 2.3.10)
- State A-G inequality and when it is equality (Theorem 2.4.1 with proof)
- Describe transition from unconstrained geometric program to its dual using A-G inequality (pages 67, 68).

Theorems and statements (Midterm 2):

- How to compute minimum norm solution of a system $A\mathbf{x} = \mathbf{b}$? *Theorem 4.1.2* - What is QR factorization of a matrix A and when it exists? *Theorem 4.1.5* - How to compute P_M ? (orthogonal projection of \mathbb{R}^m onto M) *Theorem 4.2.5*
- What is the form of solutions of underdetermined systems? *Theorem 4.3.1*
- What is the form of minimum norm solutions of underdetermined systems? *Theorem 4.3.2, with proof*

- What is the form of minimum H -norm solutions of underdetermined systems? *Theorem 4.4.2*
- What is the way of computing of the closest vector of a convex set to a given vector? *Theorem 5.1.1*
- What is the characterization of the closest vector of a convex set to a given vector using orthogonal complement? *Theorem 5.1.2*
- What is a sufficient condition for existence of a closest vector from a set C to a given vector \mathbf{x} ? *Theorem 5.1.3*
- What is a sufficient condition for existence of a unique closest vector from a set C to a given vector \mathbf{x} ? *Corollary 5.1.4*
- State basic separation theorem. *Theorem 5.1.5*
- State Support theorem. *Theorem 5.1.9*
- What can you say about $MP(z)$ if (P) is super consistent? (*Theorem 5.2.6*)
- Are there sufficient conditions for convex program (P) to have a sensitivity vector? (*Theorem 5.2.8*), *with proof* - Can MP be computed from the sensitivity vector? (*Theorem 5.2.11*), *with proof*
- State Karush-Kuhn-Tucker Theorem (Saddle point version) *Theorem 5.2.13, with proof*
- State Karush-Kuhn-Tucker Theorem (Gradient form) *Theorem 5.2.14*
- State Extended Arithmetic-Geometric Mean Inequality Include also when it is equality! *Theorem 5.3.1*
- What are sufficient condition for a constrained geometric program (GP) to have no duality gap? *Theorem 5.3.5*

Theorems and statements (Midterm 3):

- State the strong duality theorem for linear programming. *page 203*
- State duality theorem for convex programming. *Theorem 5.4.6*
- State the theorem that gives properties of Courant-Beltrami penalty function. *Theorem 6.2.3*
- What is the effect of the coercive objective function on the duality *Theorem 6.3.1* (With proof)
- For a convex program (P) , what can you tell about (P^e) ? *Theorem 6.3.2*
- When Newton's method converges in one step? *Theorem 3.1.4*
- When is Newton's method guaranteed to do decreasing steps? *Theorem 3.1.5*
- What is the special property of the steps in Steepest descent method? *Theorem 3.2.3*
- When is the Steepest descent method really a descent method? *Theorem 3.2.5*
- What is a sufficient condition for the Steepest method to converge? *Theorem 3.2.6*
- State the conditions that a good descent method should satisfy. Write them formally as well as simple explanation in English. (*page 106,107*), *C14 also why they do that they do*
- State Wolfe's Theorem about existence about descent methods. *Theorem 3.3.1*
- Describe modification of Newton's Method such that it can be used with Wolfe's Theorem. *page 114*
- What distance property is satisfied by D_{k+1} in the Boroyden's Method? *Theorem 3.4.5*

- If two vectors \mathbf{a}, \mathbf{b} have $\mathbf{a}^T \mathbf{b} > 0$, can you tell something about mapping \mathbf{a} to \mathbf{b} using a matrix? *Theorem 3.5.1*
- What distance property is satisfied by D_{k+1} in the BFGS Method? *page 126*

Definition (SDP) and Interior Point Method

- State duality theorem for Semidefinite programming
- State theorem about efficiently solving (SDP)

Other questions about stuff

- Is it true that a strictly convex function has a global minimizer? Why?
- Let x be a critical point of f and $Hf(x)$ be positive semidefinite. Is x local minimizer? Why?
- Let $f(x) = f_1(x) \cdot f_2(x)$ where both f_1 and f_2 are convex. Is f convex? Why?
- Let f be a (not strictly) concave function. Is it true that if x is a critical point and $Hf(x)$ is negative definite, then x is a local maximizer? Why? - How and why we applying QR factorization in solving least square $\mathbf{Ax} = \mathbf{b}$?
- Describe connection of Gram-Schmidt process and QR factorization.
- Is it true that for every subspace $M \subset \mathbb{R}^n$ exists a matrix A such that M is the range of A ?
- Let $M \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. How many vectors in M are closest to x . Why?
- Is it true that every two convex sets $C, D \subset \mathbb{R}^n$ can be strictly separated? That is, there exists $\mathbf{a} \in \mathbb{R}$ for every $\mathbf{c} \in C$ and $\mathbf{d} \in D$

$$\mathbf{a}^T \mathbf{c} < \mathbf{a}^T \mathbf{d}$$

- What is the relation of sensitivity vector of (P) and λ from *KKT* conditions?
- Is $MP(z)$ convex, differentiable or continuous?
- What can you tell about a program (P) if you know its sensitivity vector λ ?
- How to derive dual of a geometric program using extended AG inequality?
- Derive a dual convex program from convex program. *page 200*
- Is it true that for every convex program its optimum value is equal to the optimum value of the dual?
- Describe penalty function method
- What are differences in behavior of the Absolute value penalty function and Courant-Beltrami penalty function?
- How to modify any convex function to a coercive one? (why it is coercive?)
- What are relations between MP, MP^e, MD, MD^e ?
- By what is Newton's method approximating the function for minimization?
- Give derivation of the update matrix for D_k in Broyden's method. *page 115,116*
- Give derivation of the update matrix for D_k in BFGS method. *page 124*
- Is it true the every semidefinite program is efficiently solvable? (in polynomial time)

Extra

- express given linear program as a semidefinite program

- express given program with quadratic constraints as a semidefinite program