

Math-484 Homework #3 (convex sets and functions)

I will finish the homework before 10am Sep 19. If I spot a mathematical mistake I will let the lecturer know as soon as possible.

I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I am D14 (4 hours student).

Exercise 1: (Do I understand the definition of a convex set?)

Are the following sets D in \mathbb{R}^2 convex?

(a) $\mathbf{x} = (100, 14) \in \mathbb{R}^2, \mathbf{y} = (15, 24) \in \mathbb{R}^2$.

$D = \{\mathbf{w} \in \mathbb{R}^2 : \mathbf{w} = \lambda \mathbf{x} + (1 - \lambda)\mathbf{y}, \text{ where } 0.3 < \lambda \leq 0.7\}$

(b) $D = B((1, 0), 1) \cup (0, 0)$ recall $B(\mathbf{x}, r) = \{\mathbf{w} : \|\mathbf{x} - \mathbf{w}\| < r\}$

(c) $D = B((1, 1), 1) \cup (0, 0)$

Exercise 2: (Do I understand more than just a picture of the definition?)

Prove Theorem 2.1.4: Let $D \subseteq \mathbb{R}^n$. Then $\text{co}(D)$ coincides with the set C of all convex combinations of vectors from D .

Hint:

1) Show that C is a convex set containing D .

2) Show that if B is a convex set containing D then it also contains C .

3) Conclude that $\text{co}(D) = C$.

Exercise 3: (I recall the definition of a convex function.)

Determine whether the functions are convex, concave, strictly convex or strictly concave on the specified sets:

(a) $f(x) = \ln x$ for $x \in (0, +\infty)$

(b) $f(x) = |x|$ for $x \in \mathbb{R}$

(c) $f(x_1, x_2) = 5x_1^2 + 2x_1x_2 + x_2^2 - x_1 + 2x_2 + 3$ for $(x_1, x_2) \in \mathbb{R}^2$

(d) $f(x_1, x_2) = (x_1 + 2x_2 + 1)^8 - \ln((x_1x_2)^2)$ for $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > x_2 > 1\}$

(e) $f(x_1, x_2) = c_1x_1 + c_2/x_1 + c_3x_2 + c_4/x_2$ for $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$, where c_1, c_2, c_3 , and c_4 are positive constants

Exercise 4: (I can solve a trickier convex function problem?)

Let $f(\mathbf{x})$ be defined on set $D = \{\mathbf{x} \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\}$ as

$$f(x_1, x_2, x_3) = (x_1)^{r_1} + (x_2)^{r_2} + (x_3)^{r_3}.$$

Show that $f(\mathbf{x})$ is

(a) strictly convex on D if $r_i > 1$ for $i = 1, 2, 3$.

(b) strictly concave on D if $0 < r_i < 1$ for $i = 1, 2, 3$.

Exercise 5: (More theoretical convex functions. **D14 only**)

Prove Theorem 2.3.1: Let f be a convex function defined on an open interval $(a, b) \subset \mathbb{R}$. Then f is continuous on (a, b) .

Hint:

See page 78, exercise 4. for an outline of the proof.
