Math-484 Homework #4 (repetition and A-G inequality)

I will finish the homework before 10am Sep 26. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I am D14 (4 hours student).

1: (*A little test repetition*)
Define $f(x,y,z)$ on $\mathbb{R}^3$ as $f(x,y,z) = e^x + e^y + e^z + 2e^{-x-y-z}$. Show that $Hf(x,y,z)$ is positive definite at all points of $\mathbb{R}^3$. Find strict global minimizer of f.
Hint: You should get $(\frac{\ln 2}{4}, \frac{\ln 2}{4}, \frac{\ln 2}{4})$ *as the minimizer.*

2: (*A little test repetition*)
Show that no matter what value of a is chosen, the function $f(x_1,x_2) = x_1^3 - 3ax_1x_2 + x_2^3$ has no global maximizers. Determine the nature of the critical points of this function for all values of a.

3: (*I will recall convexity of a function*)
Show that for all positive x and y:

$$\frac{x}{4} + \frac{3y}{4} \leq \sqrt{\ln\left(\frac{e^{x^2}}{4} + \frac{3}{4}e^{y^2}\right)}$$

Hint: The desired inequality follows from convexity of an appropriate function.

4: (*Can I use $(A-G)$ inequality?*)
Solve using $(A-G)$ inequality the following problems:
a) Minimize $x^2 + y + z$ subject to $xyz = 1$ and $x,y,z > 0$
b) Maximize xyz subject to $3x + 4y + 12z = 1$ and $x,y,z > 0$
c) Minimize $3x + 4y + 12z$ subject to $xyz = 1$ and $x,y,z > 0$

5: (*I wanna be a (GP) master!*)
Solve the following (GP) where c_1, c_2, c_3 are positive numbers:
Minimize $f(x,y) = c_1x + c_2x^{-2}y^{-3} + c_3y^4$ over all $x, y > 0$.

6: (*Semidefinite matrices theoretically.* ***D14 only***)
Show that the matrix

$$A(x) = \begin{pmatrix} x^4 & x^3 & x^2 \\ x^3 & x^2 & x \\ x^2 & x & 1 \end{pmatrix}$$

is positive semidefinite for all $x \in \mathbb{R}$.
Hint: See page 79, ex. 13 and 14.