Math-484 Homework #7 (Random mix I)

I will finish the homework before 10am Oct 24. If I spot a mathematical mistake I will let the lecturer know as soon as possible.

I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I am C14 (4 hours student).

1: (*Can I work with H-norm?*)

Use minimum H-norm approach to solve the following program (P).

$$(P)\begin{cases}\text{Minimize} & f(x,y,z) = 2x^2+2xy+2y^2+z^2\\ \text{subject to} & x-y+z=3\\ & 9x+6y+2z=5\end{cases}$$

2: (*I want to know what is* epigraph.)

Let $D \subset \mathbb{R}^n$ be convex and $f : D \to \mathbb{R}$. The *epigraph* of f is a subset of $\mathbb{R}^{n+1}$ defined by

$$epi(f) = \{(\mathbf{x},\alpha) : \mathbf{x} \in D, \alpha \in \mathbb{R}, f(\mathbf{x}) \le \alpha\}.$$

a) Sketch the epigraphs of functions

$f(x) = e^x$ for $x \in \mathbb{R}$

$f(x_1,x_2) = x_1^2 + x_2^2$ for $(x_1,x_2) \in \mathbb{R}^2$

b) Show that $f(\mathbf{x})$ is convex if and only if epi(f) is convex.

c) Show that if $f(\mathbf{x})$ and $g(\mathbf{x})$ are convex functions defined on a convex set C then $h(\mathbf{x}) := \max\{f(\mathbf{x}), g(\mathbf{x})\}$ is also a convex function on C by showing that

$$\text{epi}(\max\{f(\mathbf{x}),g(\mathbf{x})\}) = \text{epi}(f(\mathbf{x})) \cap \text{epi}(g(\mathbf{x})).$$

3: (*More about orthogonal complements*)

Let M be a subspace of $\mathbb{R}^n$. Prove that the orthogonal complement $M^\perp$ of M is closed.

4: (*What is an interior?*)

Prove that if M is a subspace of $\mathbb{R}^n$ such that $M \neq \mathbb{R}^n$, then the interior M^0 of M is empty.

5: (*What are closest vectors?*)

Let C be a closed convex subset of $\mathbb{R}^n$. If $\mathbf{y} \notin C$, show that $\mathbf{x}^* \in C$ is the closest vector to $\mathbf{y}$ in C if and only if $(\mathbf{x}-\mathbf{y})^T(\mathbf{x}^*-y) \ge ||\mathbf{x}^*-\mathbf{y}||^2$ for all $\mathbf{x} \in C$.

6: (*Do I remember ancient stuff?*)

Show that

$$\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{12}+\frac{w}{12}\right)^4 \le \frac{1}{2}x^4+\frac{1}{3}y^4+\frac{1}{12}z^4+\frac{1}{12}w^4$$

holds with equality if and only if $x=y=z=w$.

7: (*Can I separate things?* ***C14 only***)
Let $C_1, C_2 \subset \mathbb{R}^n$ be both convex, $C_1^0 \neq \emptyset$ and $C_1^0 \cap C_2 = \emptyset$. Prove that there exist $0 \neq \mathbf{a} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$\mathbf{x}^T\mathbf{a} \leq \alpha \leq \mathbf{y}^T\mathbf{a}$$

for all $\mathbf{x} \in C_1$ and $\mathbf{y} \in C_2$. In other words, there is a hyperplane separating C_1 and C_2 (but both C_1 and C_2 can intersect the hyperplane).