MATH413   HW 7

due **Mar 13** before class, answer without justification will receive 0 points. Staple all your papers.

1: *(P. 257, #1)* Let $f_0, f_1, f_2, \cdots, f_n, \cdots$ denote the Fibonacci sequence. By evaluating each of the following expressions for small values of n, conjecture a general formula and then prove it, using mathematical induction and the Fibonacci recurrence:
(a) $f_1 + f_3 + f_5 + \cdots + f_{2n-1}$
(b) $f_0 + f_2 + f_4 + \cdots + f_{2n}$
(c) $f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^n f_n$
(d) $f_0^2 + f_1^2 + f_2^2 + \cdots + f_n^2$

2: *(P. 257, #3)* Prove the following about the Fibonacci numbers:
(a) f_n is even if and only if n is divisible by 3.
(b) f_n is divisible by 3 if and only if n is divisible by 4.
(c) f_n is divisible by 4 if and only if n is divisible by 6.

3: *(P. 258, #11)* The *Lucas numbers* $l_0, l_1, l_2, \ldots, ln, \ldots$ are defined using the same recurrence relation defining the Fibonacci numbers, but with different initial conditions:

$$l_n = l_{n-1} + l_{n-2}, (n \geq 2), l_0 = 2, l_1 = 1.$$

Prove that
(a) $l_n = f_{n-1} + f_{n+1}$ for $n \geq 1$
(b) $l_0^2 + l_1^2 + \ldots + l_n^2 = l_n l_{n+1} + 2$ for $n \geq 0$.

4: Find generating functions for the following sequences:
(a) $0, 0, 0, 6, -6, 6, -6, 6, -6, 6, -6, \ldots$
(b) $1, 2, 4, 1, 3, 9, 1, 4, 16, 1, 5, 25, 1, 6, 36, \ldots$
(c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \ldots$

5: For every $n \geq 0$ determine the coefficient at x^n in $(1+x)^2(1+x^2)^2(1+x^4)^2(1+x^8)^2 \cdots$ which is equal to $\prod_{i=0}^{\infty}(1+x^{2^i})^2$.

6: *(P. 259, #15)* Determine the generating function for the sequence of cubes

$$0, 1, 8, 27, \ldots, n^3, \ldots$$

7: *(P. 259, #18)* Determine the generating function for the number h_n of nonnegative integral solutions of

$$2e_1 + 5e_2 + e_3 + 7e_4 = n.$$