MATH413 **HW 11**

due **May 3** midnight. Answer without justification will receive 0 points. Submit your HW in ONE PDF file by email to Bernard Lidický. You can either type your solution or scan your handwritten solution. I really mean scan - pictures taken by your phone will not be accepted.
You do not have to do this homework. If you score on this better than on some else, it will replace the score (mean best 10 out of 11 hw counts).

1: *P. 316, # 12* Prove that the Stirlling numbers of the second $S(n, k)$ kind satisfy the following relations:
(a) $S(n, 1) = 1$ for $n \geq 1$
(b) $S(n, 2) = 2^{n-1} - 1$ for $n \geq 2$
(c) $S(n, n-1) = \binom{n}{2}$ for $n \geq 1$
(d) $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ for $n \geq 2$

2: Let $[x]_n = x \cdot (x-1) \cdot (x-2) \cdot (x-3) \cdots (x-n+1)$ and $S(n, k)$ be the Stirling number of the second kind. Show that

$$x^n = \sum_{k=0}^{n} S(n, k)[x]_k.$$

3: *P. 317, #19* Prove that the Stirling numbers of the first kind satisfy the following formulas:
(a) $|s(n, 1)| = (n-1)!$ for $n \geq 1$
(b) $|s(n, n-1)| = \binom{n}{2}$ for $n \geq 1$

4: *P.320, # 22(a)* Compute p_6 (the partition number of 6) and construct a Hasse diagram of partialy ordered set $\mathcal{P}_6$ where $\mathcal{P}_n$ contains all partitions of n (in our case $n = 6$). Suppose that $a, b \in \mathcal{P}_n$ and

$$a : n = a_1 + a_2 + \cdots + a_n \text{ where } a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$$

and

$$b : n = b_1 + b_2 + \cdots + b_n \text{ where } b_1 \geq b_2 \geq \cdots \geq b_n \geq 0.$$

Notice that we allow partitions that include 0. That is because it is easier to write formaly the following. We say that $a \leq b$ if

$$\forall i = \{1, \cdots, n\} : a_1 + \cdots + a_i \leq b_1 + \cdots + b_i.$$

So we have relation $\leq$ on $\mathcal{P}_6$ and it allows us to draw a partially ordered set of the relation. See Page 296 for more details abot $\mathcal{P}_6$.

5: *P.318, #26* Determine the conjugate partition of each of the following patitions:
(a) 12 = 5+4+2+1
(b) 15 = 6+4+3+1+1

(c) $20 = 6+6+4+4$
(d) $21 = 6+5+4+3+2+1$

6: *P.318, #27* For each integer $n > 2$, determine a self-conjugate partition of n that has at least two parts.

7: Prove that the number of partitions of n in which no part appears exactly once is equal to the number of partitions of n with no parts congruent to 1 or 5 (mod 6).
(Hint: Use generating functions for paritition number.)

8: By considering partitions with distinct (that is, non-repeated) parts, prove that

$$\prod_{k=1}^{\infty}(1+x^k) = 1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)/2}}{\prod_{k=1}^{m}(1-x^k)}$$

(Hint: look for a "maximal triangle" rather than a maximal square (Durfee square) in the Ferrers diagram).