Math-484 Homework #6 (closed sets, separation and support theorem)

I will finish the homework before 11am Oct 9. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I a 4 credits student.

1: (*Closed and open sets?*)
Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ where $f(x) > 0$ for every $x \in \mathbb{R}$. Let

$$\|f - g\| = \sup_{x \in \mathbb{R}} |f(x) - g(x)|.$$

Decide if $\mathcal{F}$ is a closed, open, or convex.

2: (*More closed and open sets?*)
Let $\mathcal{N}D$ be the set of all negative definite matrices. Decide if $\mathcal{N}D$ is closed, open, or convex.

3: (*What is an interior?*)
Prove that if M is a subspace of $\mathbb{R}^n$ such that $M \neq \mathbb{R}^n$, then the interior M^0 of M is empty.
Hint: Use that the orthogonal complement $M^\perp$ of M is also a subspace. Recall $M^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} = 0$ for all $y \in M\}$.

4: (*What are closest vectors?*)
Let C be a closed convex subset of $\mathbb{R}^n$. If $\mathbf{y} \notin C$, show that $\mathbf{x}^* \in C$ is the closest vector to $\mathbf{y}$ in C if and only if $(\mathbf{x} - \mathbf{y})^T(\mathbf{x}^* - \mathbf{y}) \geq ||\mathbf{x}^* - \mathbf{y}||^2$ for all $\mathbf{x} \in C$.

5: (*Do I remember ancient stuff?*)
Show that

$$\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{12} + \frac{w}{12}\right)^4 \leq \frac{1}{2}x^4 + \frac{1}{3}y^4 + \frac{1}{12}z^4 + \frac{1}{12}w^4$$

holds with equality if and only if $x = y = z = w$.