MATH413 FINAL

May ??? 8:00-11:00am **Name: ....**

Answer as many problems as you can. Show your work. All questions count. **An answer with no explanation will receive no credit.**

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6
Problem 7	Problem 8	Problem 9	Problem 10	Problem 11	Problem 12

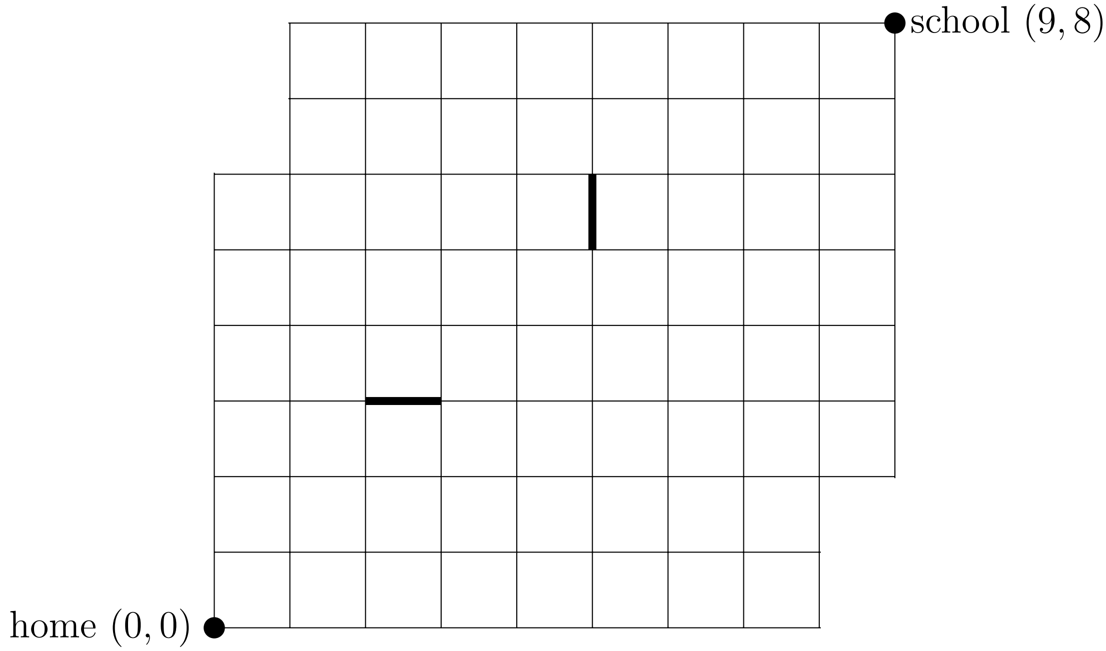
Total score
/120

Every question is for 10 points. Good luck!

1: A student has a lecture in a building located nine blocks east and eight block north of his home. Every day he walks 17 blocks to school. How many different routes are possible for him if

(a) routes must pass through **both** of the thick edges.

(b) routes must **not** pass through any of the two thick edges.



(In the picture walking is along edges*, Notice the missing corners.)*

2: There are $2n+1$ identical books to be put in a bookcase with three distinguishable shelves. In how many ways can this be done if each pair of shelves together contains more books then the other?
(Only the number of books in each shelve gives a different way of putting. Substraction principle might help)

3: Determine the number of permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 2 \cdot c\},$$

where, for each type of letter, the letters of the same type do not appear consecutively. (Thus *abbbbcaca* is not allowed but *abbbacacb* is.)

4: A basketball team has at least one game per week within 50 weeks and altogether at most 75 games. Show that there is a period of consecutive weeks during which the team had exactly 24 games.

5: Let n and k be positive integers. Give **two** proofs, one of them should be **combinatorial**, of the following identity:

$$n\binom{2n-1}{n-1} = \sum_{k=1}^{n} k\binom{n}{k}^2.$$

6: How many integer solutions exist satisfying all of the conditions below?

$$x_1 + x_2 + x_3 + x_4 = 40$$
$$5 \leq x_i \leq 15 \text{ for all } i \in \{1, 2, 3, 4\}.$$

7: **Using exponential generating series**, find the number of ways to put 30 labeled (thus distinct) people into four different rooms A, B, C, D if room A must have an even number of people (possibly 0) and the other rooms must have at least one person.

(Solution without using exponential generating series will automatically get score 0.)

8: Determine h_n, where

$$\text{for every} \quad n \geq 1 \quad h_n = 3h_{n-1} + 5n + 2, \quad h_0 = 1.$$

Check your answer for $n = 1, 2$.

9: Determine h_n, where

$$\text{for every} \quad n > 1 \quad h_n = 3h_{n-1} - 5h_{n-2}, \quad h_0 = 1, \quad h_1 = 1.$$

Check your answer for $n = 1, 2$.

10: Using the difference sequence method, find a closed formula for

$$1^4 + 2^4 + \ldots + n^4$$

.

11: A Stirling's birthday party is tonight! All $2n$ invited guests are excited to come and contribute \$5 to a present for Stirling - a very expensive bottle of wine from Catalonia (in Spain). Fibonacci volunteered to collect the contributions from all the guests. Somehow happened that n guests have exact \$5 bills but the other n guests have only \$10 bills (and want to get \$5 back). Help Fibonacci to compute in how many ways he can order collecting money from $2n$ (distinguishable!) guests such that he never runs into a trouble with giving change back.
(If Fibonacci had n-times \$5 bill then there would be $(2n)!$ *possible orderings of the guests. But he has no \$5 bill.)*

12: Prove that the Stirling numbers of the second kind satisfy the following formulas:
(a) $S(n, 2) = 2^{n-1} - 1$.
(b) $S(n, n-1) = \binom{n}{2}$.