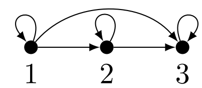
Spring 2015, MATH-201

Relations

1: How to describe “ $\leq$ ” on set $A = \{1,2,3\}$ to somebody who has no idea what is “ $\leq$ ”?

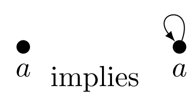
Definition of **relation** R on set A as $R \subseteq A \times A$. Notation: $(x,y) \in R$ is xRy.
Note relation is set of ordered pairs - creates *an oriented graph* for $\{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$:



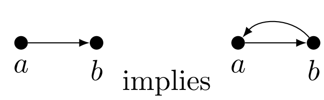
Examples of relations: $<$, $\geq$, $\equiv$, $\in$, $\subset$.
Bonus: n-ary relation instead of just binary. Also relations between sets, i.e. $R \subseteq A \times B$.
Do not confuse relation and elements being in relation and try the following question:

2: What is the number of all possible relations on $A = \{1,2,3\}$?
Let $R \subseteq A \times A$ be a relation. We call R

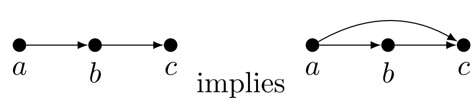
- **reflexive** if $(a,a) \in R$ for all $a \in A$



- **symmetric** if $(a,b) \in R$ implies $(b,a) \in R$ for all $a,b \in A$



- **transitive** if $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$ for all $a,b,c \in A$



3: Decide if the following relations on $\mathbb{Z}$ are reflexive, symmetric, transitive:

Relations on $\mathbb{Z}$:	$<$	$\leq$	$=$	$\mid$	$\neq$	$\nmid$	$\equiv \mod 5$
reflexive							
symmetric							
transitive							

Hint: try it on small subset, say $A = \{1,2,3,4\}$ *and draw the oriented graph.*

4: Construct relations, that are:

1. reflexive and symmetric but not transitive
2. symmetric and transitive but not reflexive

If relation is reflexive, symmetric and transitive, then we call it *equivalence*.

5: Decide if the following relations are equivalences:

1. $R \subseteq \mathbb{Z} \times \mathbb{Z}$, where xRy if "the parity of x and y are the same"
2. $R \subseteq \mathbb{R} \times \mathbb{R}$, where xRy if "$\lfloor x \rfloor$ and $\lceil y \rceil$ are equal"
3. $R \subseteq \mathbb{R} \times \mathbb{R}$, where xRy if "$|x| = |y|$"
4. $R \subseteq \mathbb{R} \times \mathbb{R}$, where xRy if "$x - \lfloor x \rfloor$ is equal to $y - \lfloor y \rfloor$"
5. $R \subseteq \mathbb{Z} \times \mathbb{Z}$, where xRy if "$x \mod 7$ is equal to $y \mod 7$" or $x \equiv y \mod 7$.
6. R is a relation on all points in the plane, where xRy means "the distance of points x and y is at most 1"
7. R is a relation on all lines in the plane, where xRy means "lines x and y intersect"
8. R is a relation on all lines in the plane, where xRy means "lines x and y do not intersect"

Let R be an equivalence relation on A. For $x \in A$ the *equivalence class* of x is all y where $(x, y) \in R$. That is $[x] = \{y : (x, y) \in R\}$.

6: Describe equivalence classes in the previous question where answer was yes.

Notice: If $R \subseteq A \times A$ is an equivalence relation, then every $a \in A$ is in exactly one equivalence class $[a]$. For every $a, b \in A$, if aRb, then $[a] = [b]$ (proof as exercise).

Let A be a set. A set $\mathcal{C}$ of subsets of A is a *partition* of A if $\cup_{C \in \mathcal{C}} = A$ and sets in $\mathcal{C}$ are pairwise disjoint.

Theorem: If R is an equivalence relation on A, then the set $\{[a] : a \in A\}$ of equivalence classes is a partition of A.

7: Work with equivalence relation $\mod 5$.

1. Describe partition $\mathbb{Z}_5$ of $\mathbb{Z}$ using equivalence relation $\mod 5$.
2. Let $x \in [2]$ and $y \in [4]$. What is the equivalence class of $x + y$ and $x \cdot y$? Does the choice of x and y matter?

Note: we can define operations $+$ and $\cdot$ on $\mathbb{Z}_5$.

8: Fill the addition and multiplication tables for $\mathbb{Z}_5$ (and $\mathbb{Z}_4$ if you have time)

+	[0]	[1]	[2]	[3]	[4]
[0]					
[1]					
[2]					
[3]					
[4]					

$\cdot$	[0]	[1]	[2]	[3]	[4]
[0]					
[1]					
[2]					
[3]					
[4]					

Definition: Let $n \in \mathbb{N}$. *Integers modulo* n is the set $\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$ equipped with $[a]+[b] = [a+b]$ and $[a] \cdot [b] = [a \cdot b]$. If n is a prime number, then $\mathbb{Z}_n$ is *finite field* of order n (inverse exists).

9: Show that $+$ and $\cdot$ for integers modulo n satisfy distributivity: $[a] \cdot ([b] + [c]) = [a] \cdot [b] + [a] \cdot [c]$.