Spring 2015, MATH-201

Facts about limits of sequences - Chapter 2.2

Squeeze lemma (Theorem about two cops): Let $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ be sequences such that $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$. Suppose $\{a_n\}$ and $\{b_n\}$ converge and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Then $\{x_n\}$ converges and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Example: Find limit of $\lim_{n\to\infty} \frac{1}{n}\sin(n)$.

Claim: Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences and $x_n \leq y_n$, for all $n \in \mathbb{N}$. Then $\lim x_n \leq \lim y_n$. Let $\lim x_n = x$ and $\lim y_n = y$. Let $\varepsilon > 0$. Goal: $x - y < \varepsilon$.

1: Let $\{x_n\}$ be convergent sequences and $x_n < y$. Is it true that $\lim x_n < y$?

Limits and algebraic operations: Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences. Then
(i) Let $\{z_n\}$, where $z_n = x_n + y_n$, converges and $\lim(x_n + y_n) = \lim z_n = \lim x_n + \lim y_n$.
(ii) Let $\{z_n\}$, where $z_n = x_n - y_n$, converges and $\lim(x_n - y_n) = \lim z_n = \lim x_n - \lim y_n$.
(iii) Let $\{z_n\}$, where $z_n = x_n y_n$, converges and $\lim(x_n y_n) = \lim z_n = (\lim x_n) \cdot (\lim y_n)$.
(iv) If $\lim y_n \neq 0$ and $y_n \neq 0$ forall $n \in \mathbb{N}$, then the sequence $\{z_n\}$,where $z_n := x_n/y_n$, converges and

$$\lim \frac{x_n}{y_n} = \lim z_n = \frac{\lim x_n}{\lim y_n}.$$

Proof of (iii). We want $\lim(x_n y_n) = xy$. We want $|x_n y_n - xy| < \varepsilon$ for large n. Trick: $|x_n y_n - xy| = |x_n y_n - (x + x_n - x_n)y| = |x_n(y_n - y) - (x - x_n)y|$

2: Prove (i)

Convergence tests

3: Let $\{x_n\}$ be a sequence. Suppose there is an $x \in \mathbb{R}$ and a convergent sequence $\{a_n\}$ such that $\lim a_n = 0$ and $|x_n - x| \leq a_n$ for all n. Show that $\{x_n\}$ converges and $\lim x_n = x$.
Example: $x_n = 3 + \frac{1}{n}$, $x = 3$, $a_n = \frac{1}{n}$ Hint: Verify the definition of a limit.

Ratio test for sequences. Let $\{x_n\}$ be a sequence such that $x_n \neq 0$ for all n and such that the limit

$$L = \lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|}$$

exists. Then
(i) If $L < 1$, then $\{x_n\}$ converges and $\lim x_n = 0$.
(ii) If $L > 1$, then $\{x_n\}$ is unbounded (hence diverges).

4: Use ration test to show that $\{\frac{3^n}{n!}\}$ converges.

5: Find an example of a sequence $\{x_n\}$ where ratio test does not decide if $\{x_n\}$ converges.

6: Prove the ratio test. Outline: Notice that $|x_n| = \frac{|x_n|}{|x_{n-1}|} \cdot \frac{|x_{n-1}|}{|x_{n-2}|} \cdots |x_k|$, where $k < n$. And then for large n (and k), one can bound $|x_n|$ using $r^{n-k} x_k$, where r can come from L, which is the limit of $\frac{|x_n|}{|x_{n-1}|}$. Then you need two cases (i) and (ii).