Spring 2015, MATH-201

Chapter 2.5 Series - 2.5.1, 2.5.2, 2.5.3

Series: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Then

$$\sum_{n=1}^{\infty} x_n$$

is a *series*. Let $s_n = \sum_{k=1}^{n} x_k$ be a *partial sum*. We use

$$\sum_{n=1}^{\infty} x_n = x \text{ if } \lim_{n\to\infty} s_n = x.$$

$\sum_{n=1}^{\infty} x_n$ is *convergent* or *divergent* if $\{s_n\}_{n=1}^{\infty}$ is convergent or divergent.
Example: $\sum_{n=1}^{\infty}(-1)^n$ is not convergent.

Example: $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. We use $1 - \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2^n}$ and show $|1 - s_n| < \varepsilon$.

1: Show that $\sum_{n=1}^{\infty} \sqrt[n]{\frac{1}{100}}$ is divergent. Hints: What is x_n and $\lim x_n$? Use $\sqrt[n]{\frac{1}{100}} \geq \frac{1}{100}$.

2: Let $0 < r < 1$. Show that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. Hint: Use partial sums $\sum_{k=0}^{n} r^k = \frac{1-r^n}{1-r}$.

3: Let $M \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} x_n$ is convergent iff $\sum_{n=M}^{\infty} x_n$ is convergent.
Hint: Consider the difference of $\sum_{n=1}^{k} x_n$ and $\sum_{n=M}^{k} x_n$ if $k > n$.

Recall: A sequence $\{s_n\}$ is a *Cauchy sequence* if $\forall \varepsilon > 0, \exists M \in \mathbb{N}, \forall n \geq M, \forall k \geq M, |s_n - s_k| < \varepsilon$.

A series $\sum_{n=1}^{\infty} x_n$ is called *Cauchy* or a *Cauchy series*, if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence. (recall $\{s_n\}_{n=1}^{\infty}$ is Cauchy iff $\{s_n\}_{n=1}^{\infty}$ is convergent)

When is $\sum_{n=1}^{\infty} x_n$ Cauchy? Fix $\varepsilon > 0$. Since s_n is Cauchy, $\exists M \in \mathbb{N}, \forall n \geq M, \forall k \geq M, |s_n - s_k| < \varepsilon$...

The series $\sum_{n=1}^{\infty} x_n$ is Cauchy if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for every $n \geq M$ and every $k > n$ we have

$$\left| \sum_{i=n+1}^{k} x_i \right| \leq \varepsilon.$$

4: Let $\sum_{n=1}^{\infty} x_n$ be a convergent series. Show that sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n = 0$.
Hint: Use that $\sum_{n=1}^{\infty} x_n$ is Cauchy. Hint 2: Look 2 lines up.

5: Find a lower bound for $\sum_{n=k+1}^{2k} \frac{1}{n}$. Hint: try lower bound $\frac{1}{2}$, try for $k = 3$ (or $k = 4$).

6: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by showing it is not Cauchy.
Hint 1: By contradiction, suppose it is Cauchy. Then for $\varepsilon = \frac{1}{4}$ exists M such that Cauchy condition leads to contradiction
Hint 2: Use previous question.

7: Let $a \in \mathbb{R}$ and $\sum x_n$ be convergent. Show that $\sum(a \cdot x_n)$ is convergent and $\sum(a \cdot x_n) = a \sum x_n$.
Hint V1: Check the Cauchy condition. Use that a is a constant and hence εa can be arbitrarily small.
Hint V2: Check the sum condition. Use linearity of sum and lim.