Spring 2015, MATH-201

Chapters 2.5.4, 2.6.3 - Absolute convergence

Consider $\{a_n\}_{n=1}^{\infty} = (\frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots)$. Note $a_{2n-1} = \frac{1}{n}$ and $a_{2n} = -\frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} a_n =$$

Let $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$. Then $\{a_n\}_{n=1}^{\infty} = (x_1, y_1, x_2, y_2, x_3, \ldots)$. Recall

$$\sum_{n=1}^{\infty} x_n = \qquad\qquad \sum_{n=1}^{\infty} y_n =$$

Consider

$$\{b_n\}_{n=1}^{\infty} = \left(\frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, -\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, -\frac{1}{3}, \frac{1}{8}, \ldots, \frac{1}{15}, -\frac{1}{4}, \frac{1}{16} \ldots, \frac{1}{31}, -\frac{1}{5}, \ldots\right).$$

Notice that $\cup_n\{a_n\} = \cup_n\{b_n\}$, both sequences have the same numbers, b_n is just a *rearranging* of a_n. But

$$\sum_{n=1}^{\infty} b_n =$$

1: Let $\{c_n\}_{n=1}^{\infty}$ be a rearranging of $\{a_n\}_{n=1}^{\infty}$. What are possibilities for $\sum_{n=1}^{\infty} c_n =$?

Let $\sum_{n=1}^{\infty} z_n = z$ be a series. How to guarantee every rearranging of z_n sums to z?

The series $\sum_{n=1}^{\infty} z_n$ *converges absolutely* if $\sum_{n=1}^{\infty} |z_n|$ converges.
Theorem: If $\sum_{n=1}^{\infty} z_n$ converges absolutely and $\sum_{n=1}^{\infty} z_n = z$. Then any rearrangement of $\sum_{n=1}^{\infty} z_n$ converges to z and it is absolutely convergent.
If $\sum_{n=1}^{\infty} z_n$ converges but not absolutely, we call $\sum_{n=1}^{\infty} z_n$ *conditionally convergent* (order matters).
Example: $\sum \frac{(-1)^n}{2^n}$ is absolutely convergent.

2: Is $\sum_{n=1}^{\infty} a_n$ absolutely convergent?

3: Let $\sum x_n$ be absolutely convergent. Show that $\sum x_n$ is convergent.
Hint1: Use that $\sum |x_n|$ is Cauchy and conclude $\sum x_n$ is Cauchy.
Hint2: Use triangle inequality.
$\sum |x_n|$ is Cauchy if $\forall \varepsilon > 0, \exists M, \forall$

$\sum x_n$ is Cauchy if $\forall \varepsilon > 0, \exists M, \forall$

Chapters 2.5.5, 2.5.6 - Convergence tests

Let $\sum x_n$ and $\sum y_n$ be series such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$.
(i) If $\sum y_n$ converges, then so does $\sum x_n$. (ii) If $\sum x_n$ diverges, then so does $\sum y_n$.
(The partial sums are monotone increasing, monotone increasing sequence converges iff it is bounded.)

Proposition 2.5.15 (p-series or the p-test). For $p \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.
Example: $\sum \frac{1}{n^2+1}$

Example: $\sum \frac{1}{\sqrt{n}}$

Proposition 2.5.17 (Ratio test). Let $\sum x_n$ be a series such that $L = \lim_{n\to\infty} \frac{|x_n+1|}{|x_n|}$ exists. Then
(i) If $L > 1$, then $\sum x_n$ diverges.
(ii) If $L < 1$, then $\sum x_n$ converges absolutely.

Example: Series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges absolutely.

Proof of Ratio test:
Let $L = \lim_{n\to\infty} \frac{|x_n+1|}{|x_n|}$
(i) $L > 1$: Use ratio test for sequences and show $|x_n|$ is getting large.

(ii) $L < 1$: Goal: $\sum |x_n|$ converges. Enough to show that $\sum |x_n|$ is bounded.
Find upper bound on $\sum_{n=M}^{\infty} |x_n|$ using that for $0 < r < 1$ the geometric series $\sum_{k=0}^{\infty} r^k$ converges.
Use similar idea as Ratio test for sequences how to write x_n using previous for large n.

4: Decide if $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$ converges absolutely.