Spring 2015, MATH-201

Chapters 2.6.1 - Root test

Proposition 2.6.1 (Root test). Let $\sum_{n=1}^{\infty} x_n$ be a series and let

$$L = \limsup_{n\to\infty} |x_n|^{\frac{1}{n}}.$$

(i) If $L > 1$ then $\sum x_n$ diverges. (ii) If $L < 1$ then $\sum x_n$ converges absolutely.

Example: $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

Example: $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

Proof of the Root test:

Case 1: $L > 1$ goal: $\sum_{n=1}^{\infty} |x_n|$ contains infinitely many entries bigger than 1.

Case 2: $L < 1$: Use that $|x_n| < r^n$ if n large and $L < r < 1$. Then bound the tail of $\sum |x_n|$ using geometric series $\sum r^n$.

1: Find $\sum x_n$ that is absolutely convergent and $L = 1$.

2: Find $\sum x_n$ that is divergent and but $L = 1$.

3: Find $\sum x_n$ that is convergent (but not absolutely convergent) and $L = 1$.