## Chapter 3.2 Pigeonhole principle - strong form

1: Suppose grades $A, B, C, D$ and $F$ are assigned. What is the smallest size of class to ensure that at least one of the following happens:

$$
5 \times A \text { or } 5 \times B \text { or } 4 \times C \text { or } 2 \times D \text { or } 1 \times F
$$

## Theorem 3.2.1

Let $q_{1}, q_{2}, \ldots, q_{n} \in \mathbb{Z}^{+}$. If $\left(\sum_{i=1}^{n} q_{i}\right)-n+1$ objects are placed to $n$ boxes then exists $j$ such that box $j$ contains at least $q_{j}$ objets.

- if $q_{j}=2$ for all $j$, then it is ordinary pigeonhole principle
- if $q_{j}=r$ for all $j$, then the theorem says if there are $(r-1) n+1$ objects, then at least one box contains $r$ objects
same as: if average of integers $q_{1}, \ldots, q_{n}$ is more than $r-1$, there is one greater or equal to $r$.
- maybe think of $\left(\sum_{i=1}^{n} q_{i}\right)-n+1$ as $\left(\sum_{i=1}^{n} q_{i}-1\right)+1$

2: Proof Theorem 3.2.1 (by constradiction)

3: Two disks, one smaller than the other, are each divided into 200 congruent sectors. In the larger disk, 100 of the sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk, each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The small disk is then placed on the larger disk so that their centers coincide. Show that it is possible to align the two disks so that the number of sectors of the small disk whose color matches the corresponding sector of the large disk is at least 100 .


Theorem (Erdős-Szekeres) Every sequence $A=a_{1}, a_{2}, \ldots, a_{n^{2}+1}$ of real number contains either increasing or decreasing subsequene if length $n_{1}$.

$$
a_{k_{1}} \geq a_{k_{2}} \geq \cdots \geq a_{k_{n+1}} \text { or } a_{k_{1}} \leq a_{k_{2}} \leq \cdots \leq a_{k_{n+1}}
$$

where

$$
k_{1}<k_{2}<\cdots<k_{n+1}
$$

Proof: Assume $A$ does not contain increasing subsequence of length $n+1$. Let $L(i)$ be the length of longest increasing subsequence starting at $a_{i}$. Note that $1 \leq L(i) \leq n$.

4: Use pigeonhole principle on $\{L(i)\}$ and finish the proof. What can you say about $a_{i}$ and $a_{j}$ if $L(i)=L(j)$ and $i<j$ ?

Chinese Remainder Theorem Let $m, n$ be relatively prime (no common divisor). Let $0 \leq a \leq m-1$ and $0 \leq b \leq n-1$, where $a, b \in \mathbb{Z}$. Then there exists $x \in \mathbb{Z}$, such that both

$$
x \quad \bmod m=a \quad x \quad \bmod n=b
$$

(Recall mod stands for modulo.)
Proof: We want to find $x \in \mathbb{Z}$ such that

$$
x=p \cdot m+a \quad x=q \cdot n+b,
$$

where $p$ and $q$ are some integers. Consider numbers

$$
X=\{a, m+a, 2 m+a, 3 m+a, \ldots,(n-1) m+a\} .
$$

They all satisfy $\bmod m=a$.
5: What can you say if there exists $y, z \in X$ such that $y \bmod n=z \bmod n$ ? (Hint: get contradiction, show it is not possible, try their difference)
Use pigeonhole principle to finish the proof.

