# Practice problems for Calculus I

We can find the derivative of the inverse of a function, even if we can't find the inverse!

1 *Example:* For $f(x) = e^{2-x} - 4x + 10$, let $g(x) = f^{-1}(x)$. Find $g'(3)$.

2 *Example:* Suppose that $h(x) = f(x)g(x)$ is an invertible differentiable function. Further suppose that we know the following information:

| | x=0 | x=3 | x=7 | x=11 | x=16 |
|---|---|---|---|---|---|
| $f(x)$ | 3 | 4 | 10 | 24 | 66 |
| $f'(x)$ | 1 | 6 | 7 | 19 | 28 |
| $g(x)$ | 5 | 4 | 3 | 2 | 1 |
| $g'(x)$ | −1 | −1 | −1 | −2 | −2 |

Find the tangent line to the curve $y = h^{-1}(x)$ at $x = 16$.

For derivatives with lots of multiplications and/or a function in the exponent, can use logs.

3 *Example:* Find $\dfrac{d}{dx}\left(x^{(e^x)}\right)$.

4 *Example:* Find $\dfrac{d}{dx}\left(\dfrac{\arctan(x)\cos(x)}{(x^2+1)e^x}\right)$

If two (or more) variables are related, then their rates of change are related.

5 *Example:* A train is heading east towards Conjunction Junction at a speed of 15 miles and is currently 25 miles away, at the same time a second train is heading south away from Conjunction Junction at a speed of 10 miles and is currently 60 miles away. At what rate are the two trains moving apart from each other?

6 *Example:* If applying a force to a wrench at a 90° angle the amount of torque $\tau$ produced is $\tau = rF$ where $r$ is the length along the wrench you apply the force and $F$ is the amount of force applied. Suppose that you need to produce a constant torque of 50 Newton meters. Find $dr/dt$ if the amount of force you can generate is changing at a rate of $dF/dt = -5$ Newtons per second when $F = 100$ Newtons.

Tangent lines are good (linear) approximations to functions and can be used for estimation.

7 *Example:* Find the linearization of $f(x) = \sqrt[4]{x}$ at $x = 625$. Use this to estimate $\sqrt[4]{610}$.

8 *Example:* Find the linearization of $f(t) = \dfrac{5t}{t^2+1}$ at $t = 2$. Use this to estimate $f(2.05)$.

Find max/min of continuous function, $f(x)$, on $a \leqslant x \leqslant b$ by evaluating $f$ at *all* critical points.

9 *Example:* Find the absolute minimum and absolute maximum of $h(s) = 2s^3 + 3s^2 - 12s - 18$ for $-3 \leqslant s \leqslant 2$.

10 *Example:* Find the absolute minimum and absolute maximum of $g(t) = (t-3)e^t - (3/2)t(t-4)$ for $1 \leqslant t \leqslant 3$

First derivative tells increasing/decreasing
Second derivative tells concave up/concave down

11 *Example:* Find all inflection points of $y = x^{8/3} - 8x^{5/3}$. Justify your answer.

12 *Example:* For $g(x) = 4x^5 - 5x^4 - 40x^3 + 137$, find the maximal interval(s) where the function is increasing and decreasing.

We can use derivatives to find max/min.
If we don't have a function, we make one!

13 *Example:* Find and classify all critical points for $h(t) = (2t+3)e^{t^2}$.

14 *Example:* Find the maximum area of a rectangle which has the bottom on the x-axis and the top vertices lying on the curve $y = e^{-x^2}$.

Newton's method approximates roots

15 *Example:* Use Newton's method to approximate a solution to $\cos(x) = x^2$. Starting with $x_0 = 1$, find $x_2$ (give answer to four decimal points).

16 *Example:* Use Newton's method to approximate a root to $y = x^3 - 3x^2 + 6x - 3$ starting with $x_0 = 1$ give the exact value of $x_2$.

Basic properties of integrals can help to simplify/rewrite expressions.

17 *Example:* Simplify the following to a single integral:

$$\int_1^6 f(t)\,dt + \int_1^4 f(t)\,dt + 2\int_2^1 f(t)\,dt + \int_6^4 f(t)\,dt.$$

18 *Example:* Find $\displaystyle\int_{-5}^{5} \left(3 + t^7\cos(t)\right) dt$.

To find antiderivatives, we often first rewrite.

19 *Example:* Find $\displaystyle\int \frac{1}{\left(\sin(\frac{1}{2}\theta) - \cos(\frac{1}{2}\theta)\right)^2}\, d\theta$.

20 *Example:* Find $\displaystyle\int \frac{(x-1)^2 + 2}{x}\, dx$.

We can approximate the "total" of a function by using Riemann sums.

21 *Example:* Approximate the area under the curve $y = x^2 - 3x + 5$ from $x = 0$ to $x = 4$ using a Riemann sum with four equally spaced intervals and using the *right* endpoint of each interval.

22 *Example:* Given that $v(t) = \dfrac{5x}{x^2+1}$ is a velocity of the particle, estimate the distance it has traveled from $t = 0$ to $t = 4$ by using a Riemann sum with four equally spaced intervals and using the *left* endpoints of each interval.

Some integrals can be found using geoemetry.

23 *Example:* Find the area between $y = \sqrt{1-x^2}$ and $y = 2|x| - 2$ for $-1 \leqslant x \leqslant 1$.

24 *Example:* Find $\displaystyle\int_0^8 h(x)\,dx$ where

$$h(x) = \begin{cases} 2 - |x-2| & \text{if } 0 \leqslant x \leqslant 3; \\ 3 - |x-5| & \text{if } 3 \leqslant x \leqslant 8. \end{cases}$$

The Fundamental Theorem of Calculus connects integration and differentiation.

25 *Example:* For $x > 0$, find $f(x)$ given that

$$\int_x^2 tf(t)\,dt = e^{3x-6} + C\cos(\pi x) + 2x + 1.$$

(Note that you will have to solve for "C".)

26 *Example:* Find $\dfrac{d}{dx}\left(\displaystyle\int_x^{x^2} \frac{\arctan t}{t^4+1}\,dt\right)$.

Substitution is one of the most important tools available for working with integrals.

27 *Example:* Evaluate $\displaystyle\int_1^3 \frac{dy}{y^{1/2} + y^{3/2}}$ by substituting $u = y^{1/2}$.

28 *Example:* Find $\displaystyle\int x\left(\sin(x^2) + \cos(x^2)\right)^2 \sin(x^2)\,dx$.

Integration can be used to find areas/averages.

29 *Example:* Find the area bounded by $f(x) = 2\sin x$ and $g(x) = \sec x \tan x$ between $x = 0$ and the smallest $x > 0$ where the two curves intersect.

30 *Example:* Find the average value of the function $f(x) = \dfrac{x}{x^2+1}$ for $0 \leqslant x \leqslant 2$.

This is how to solve separable differential equations: Separate, Integrate, Uncomplicate

31 *Example:* Find $y(t)$ if $y' = \dfrac{\cos(t)}{2y+2}$ and $y(0) = -2$.

32 *Example:* The number of acres of a forest after t centuries, denoted $A(t)$, grows faster with time and also is curtailed by its size (i.e., a larger forest exhibits slower growth). We can model this as $A' = \dfrac{26t}{\sqrt{A}}$. Given that the forest initially (i.e., at $t = 0$) has nine acres, how many acres will it have after six centuries (i.e., at $t = 6$)?

For indeterminate limits (i.e., $\frac{0}{0}$) use L'Hospital.

33 *Example:* Find $\displaystyle\lim_{t\to 0} \frac{\int_{3t}^{5t} \sin(x)\,dx}{t^2}$.

34 *Example:* Given that f is twice differentiable and $\displaystyle\lim_{x\to 2} \frac{xf(x)+4}{(x-2)^2} = 3$, find $f(2)$, $f'(2)$, and $f''(2)$. (Hint: if denominator goes to 0, then the only possible chance for a limit to exist is for it to be indeterminate and push on with L'Hospital.)

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Basic integrals you are expected to know:

$$\int u^n\,du = \frac{1}{n+1}u^{n+1} + C \qquad \int \frac{du}{u} = \ln(u) + C$$

$$\int e^u\,du = e^u + C \qquad \int \frac{du}{1+u^2} = \arctan(u) + C$$

$$\int \sin(u)\,du = -\cos(u) + C$$

$$\int \cos(u)\,du = \sin(u) + C$$

$$\int \sec^2(u)\,du = \tan(u) + C$$

$$\int \sec(u)\tan(u)\,du = \sec(u) + C$$

(If the integral you are doing is *not* one of the above, then don't try to do it (you will likely be wrong)!)

Trig identities you are expected to know:

$$\sin^2(u) + \cos^2(u) = 1 \qquad \sin(2u) = 2\sin(u)\cos(u)$$

Make sure to pay attention to algebra, arithmetic, and even just basic copying. Most points are lost for "simple" mistakes that then snowball. The ability to rewrite expressions, to factor, to expand, etc. are the lifeblood of a good calculus practitioner. Similarly know the basic rules, most problems are usually testing on some basic rules which have been slightly obfuscated.

① $g'(3) = \frac{1}{f'(f^{-1}(3))}$

$e^{2-x} - 4x + 10 = 3$

↑

guess $x=2$ to make this nice

$= \frac{1}{f'(2)}$

$e^{0} - 4\cdot 2 + 10 = 1 - 8 + 10 = 3$ ✓

so $f^{-1}(3) = 2$

$f'(x) = -e^{2-x} - 4$

$= \frac{1}{-1-4}$

$= \boxed{\frac{-1}{5}}$

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② $y = h^{-1}(x)$ tangent line at $x=16$

need $h^{-1}(16)$ and $(h^{-1})'(16)$

$h(x) = f(x)g(x) = 16$ @ $x=3$ $f(3)=4, g(3)=4$

so $h(3) = 4\cdot 4 = 16$

$\Rightarrow h^{-1}(16) = 3$

$(h^{-1})'(16) = \frac{1}{h'(h^{-1}(16))}$

$= \frac{1}{h'(3)}$

$= \boxed{\frac{1}{20}}$

note $h(x) = f(x)g(x)$

$h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(3) = f'(3)g(3) + f(3)g'(3)$

$= 6\cdot 4 + 4(-1)$

$= 24 - 4 = 20$

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③ $y = x^{(e^x)}$ ⇝ $\ln y = e^x \ln x$ ⇝ $\frac{1}{y}\frac{dy}{dx} = e^x \ln x + e^x \cdot \frac{1}{x}$

⇝ $\frac{dy}{dx} = \left(e^x \ln x + \frac{e^x}{x}\right) y$

$\boxed{= \left(e^x \ln x + \frac{e^x}{x}\right) x^{(e^x)}}$

(4) $y = \dfrac{\arctan(x)\cos(x)}{(x^2+1)e^x} \rightsquigarrow \ln y = \ln\left(\dfrac{\arctan(x)\cos(x)}{(x^2+1)e^x}\right)$

$$= \ln(\arctan(x)) + \ln(\cos(x)) - \ln(x^2+1) - \ln(e^x)$$

$$= \ln(\arctan(x)) + \ln(\cos x) - \ln(x^2+1) - x$$

$$\rightsquigarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{\arctan x} \cdot \frac{1}{x^2+1} + \frac{1}{\cos x} \cdot (-\sin x) - \frac{1}{x^2+1} \cdot 2x - 1$$

$$\rightsquigarrow \frac{dy}{dx} = \left(\frac{1}{(x^2+1)\arctan x} - \tan x - \frac{2x}{x^2+1} - 1\right) y$$

$$\boxed{= \left(\frac{1}{(x^2+1)\arctan x} - \tan x - \frac{2x}{x^2+1} - 1\right)\frac{\arctan(x)\cos(x)}{(x^2+1)e^x}}$$

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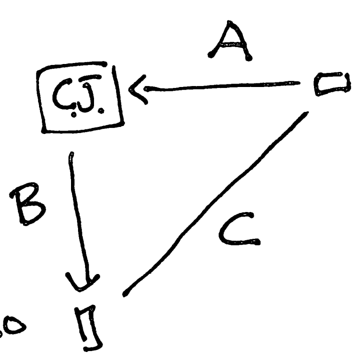
(5)




$B = 60$

$\frac{dB}{dt} = 10$

$A = 25$

$\frac{dA}{dt} = -15$

$C^2 = 25^2 + 60^2$

$= 4225$

$C = \sqrt{4225} = 65$

$$A^2 + B^2 = C^2$$

$$\cancel{2}A\frac{dA}{dt} + \cancel{2}B\frac{dB}{dt} = \cancel{2}C\frac{dC}{dt}$$

$$25 \cdot (-15) + 60 \cdot 10 = 65\frac{dC}{dt}$$

$$225 = 65\frac{dC}{dt}$$

$$\frac{dC}{dt} = \frac{225}{65} = \boxed{\frac{45}{13}}$$

(6) $\tau = rF \quad \Rightarrow \quad 0 = \underset{100}{\frac{dr}{dt}} F + \underset{\frac{1}{2}}{r} \underset{-5}{\frac{dF}{dt}}$

$\tau$: torque which we want fixed at 50

$50 = r \cdot 100$
$\rightsquigarrow r = \frac{1}{2}$
at our time

$100 \frac{dr}{dt} = \frac{5}{2}$

$\frac{dr}{dt} = \frac{5}{200} = \boxed{\frac{1}{40}}$

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(7) $f(x) = x^{1/4}$ @ $a = 625$

$f(625) = 625^{1/4} = 5$

$f'(x) = \frac{1}{4} x^{-3/4} = \frac{1}{4(x^{1/4})^3}$

$f'(625) = \frac{1}{4 \cdot 5^3} = \frac{1}{500}$

$L(x) = f(a) + f'(a)(x-a)$

$= 5 + \frac{1}{500}(x - 625)$

$(610)^{1/4} = f(610) \approx L(610) = 5 + \frac{1}{500}(610-625) = 5 - \frac{15}{500}$

$= 5 - \frac{30}{1000}$

$= \boxed{4.97}$

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(8) $f(t) = \frac{5t}{t^2+1}$ @ $t = 2$

$f(2) = \frac{5 \cdot 2}{2^2+1} = 2$

$f'(t) = \frac{(t^2+1) \cdot 5 - 5t \cdot 2t}{(t^2+1)^2}$

$f'(2) = \frac{25 - 40}{25} = \frac{-15}{25} = -\frac{3}{5}$

$L(t) = f(a) + f'(a)(t-a)$

$= 2 - \frac{3}{5}(t-2)$

$f(2.05) \approx L(2.05) = 2 - \frac{3}{5}(2.05-2) = 2 - \frac{3}{5}(.05) = 2 - .03$

$= \boxed{1.97}$

(9) $h(s) = 2s^3 + 3s^2 - 12s - 18$ $\qquad -3 \le s \le 2$

$h'(s) = 6s^2 + 6s - 12 = 6(s^2 + s - 2) = 6(s+2)(s-1)$ ← $-2$ and $1$ both in interval $-3 \le s \le 2$. Derivative never undefined

Critical points

bdy $\begin{cases} -3 \\ 2 \end{cases}$

$f' = 0$ $\begin{cases} -2 \\ 1 \end{cases}$

$f(-3) = -9$

$f(2) = -14$

$\boxed{\begin{array}{l} f(-2) = 2 \quad \leftarrow \text{absolute max} \\ f(1) = -25 \quad \leftarrow \text{absolute min} \end{array}}$

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(10) $g(t) = te^t - 3e^t - \frac{3}{2}t^2 + 6t$

$g'(t) = e^t + te^t - 3e^t - 3t + 6$

$= te^t - 2e^t - 3t + 6$

$= (t-2)e^t - 3(t-2)$

$= (t-2)(e^t - 3)$ ← never undefined; $g' = 0$ if $t = 2$ or $e^t = 3$, i.e. $t = \ln 3 \approx 1.098$

Critical pts

bds $\begin{cases} 1 \\ 3 \end{cases}$

$g' = 0$ $\begin{cases} 2 \\ \ln 3 \end{cases}$

$g(1) = \frac{9}{2} - 2e \approx -.936$

$g(3) = \frac{9}{2} = 4.5$

$g(2) = 6 - e^2 \approx -1.389$

$g(\ln 3) \approx -.9229$

$\boxed{\begin{array}{l} g(3) = \frac{9}{2} \leftarrow \text{absolute max} \\ g(2) = 6 - e^2 \leftarrow \text{absolute min} \end{array}}$

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(11) $y = x^{8/3} - 8x^{5/3}$

$y' = \frac{8}{3}x^{5/3} - \frac{40}{3}x^{2/3}$

$y'' = \frac{40}{9}x^{2/3} - \frac{80}{9}x^{-1/3}$

$= \frac{40}{9}x^{-1/3}(x - 2)$

$x = 0$ undefined $\qquad x = 2$ $y'' = 0$

$f''$: 0, 2

$f'' > 0$ concave up $\quad$ $f'' < 0$ concave down $\quad$ $f'' > 0$ concave up

inflection points are where concavity changes which happens at $\boxed{x = 0 \text{ and } x = 2}$

(12) $g(x) = 4x^5 - 5x^4 - 40x^3 + 137$

$g'(x) = 20x^4 - 20x^3 - 120x^2$

$= 20x^2(x^2 - x - 6)$

$= 20x^2(x-3)(x+2)$ ← zeroes at $-2, 0, 3$

$g'$: $-2$, $0$, $3$

$g'>0$ increasing | $g'<0$ decreasing | $g'<0$ decreasing | $g'>0$ increasing

$\underbrace{\text{decreasing} \quad \text{decreasing}}_{\text{decreasing over whole interval}}$

$$\boxed{\begin{array}{l} \text{increasing:} \\ \quad x \le -2 \text{ and } x \ge 3 \\ \text{decreasing:} \\ \quad -2 \le x \le 3 \end{array}}$$

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(13) $h'(t) = 2e^{t^2} + (2t+3)e^{t^2} \cdot 2t = (2 + 4t^2 + 6t)e^{t^2} = 2(2t^2 + 3t + 1)e^{t^2}$

$= 2(2t+1)(t+1)\underbrace{e^{t^2}}_{>0}$  critical pts at $t = -\frac{1}{2}$, $t = -1$

First derivative test: $h'$

$h'>0$ $\boxed{-1}$ $h'<0$ $\boxed{-\frac{1}{2}}$ $h'>0$

$$\boxed{\begin{array}{ll} x=-1 \text{ is local max} & x=-\frac{1}{2} \text{ is local min} \end{array}}$$

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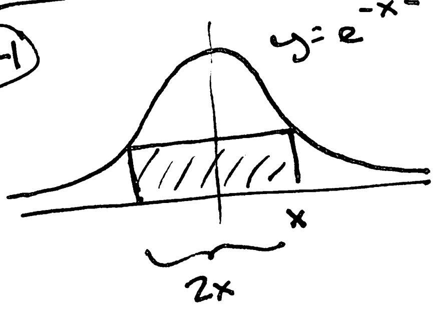
(14)




Area $= A(x) = 2xe^{-x^2}$

$A'(x) = 2e^{-x^2} + 2xe^{-x^2} \cdot (-2x)$

$= 2\underbrace{e^{-x^2}}_{>0}(1 - 2x^2) = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \sqrt{\frac{1}{2}}$

$A'$: $A'>0$ $\sqrt{\frac{1}{2}}$ $A'<0$

by $1^{st}$ derivative test this is a max

max area

$= A(\sqrt{\tfrac{1}{2}})$

$= 2\sqrt{\tfrac{1}{2}}\, e^{-(\sqrt{\frac{1}{2}})^2} = \sqrt{2}\, e^{-1/2}$

$= \boxed{\sqrt{\frac{2}{e}}}$

(15) $\cos(x) = x^2 \rightsquigarrow$ want root of $f(x) = \cos(x) - x^2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos(x_n) - x_n^2}{-\sin(x_n) - 2x_n} = x_n + \frac{\cos(x_n) - x_n^2}{\sin(x_n) + 2x_n}$$

$x_0 = 1$

$$x_1 = 1 + \frac{\cos(1) - 1}{\sin(1) + 2} \approx 0.8382184$$

$$x_2 = 0.8382184 + \frac{\cos(0.8382184) - (0.8382184)^2}{\sin(0.8382184) + 2(0.8382184)} \approx 0.824785$$

$$\boxed{0.8247}$$

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(16) $f(x) = x^3 - 3x^2 + 6x - 3$

$f'(x) = 3x^2 - 6x + 6$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n^2 + 6x_n - 3}{3x_n^2 - 6x_n + 6}$$

$x_0 = 1$

$$x_1 = 1 - \frac{1-3+6-3}{3-6+6} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$x_2 = \frac{2}{3} - \frac{(\frac{2}{3})^3 - 3(\frac{2}{3})^2 + 6(\frac{2}{3}) - 3}{3(\frac{2}{3})^2 - 6(\frac{2}{3}) + 6} = \boxed{\frac{61}{90}}$$

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(17) $$\int_1^6 f(t)\,dt + \int_6^4 f(t)\,dt = \int_1^4 f(t)\,dt$$

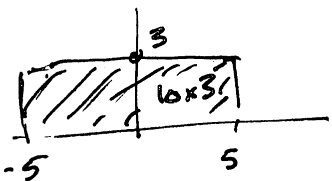
$$\Rightarrow \int_1^6 f(t)\,dt + \int_6^4 f(t)\,dt + \int_1^4 f(t)\,dt + 2\int_2^1 f(t)\,dt = 2\int_1^4 f(t)\,dt + 2\int_2^1 f(t)\,dt$$

$$= 2\int_2^4 f(t)\,dt$$

Used $\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt$

$$= \boxed{\int_2^4 2f(t)\,dt}$$

(18) $$\int_{-5}^{5}\left(3+t^{7}\cos(t)\right)dt = \int_{-5}^{5}3\,dt + \underbrace{\int_{-5}^{5}\underbrace{t^{7}\cos(t)}_{\text{odd}}\,dt}_{=0,\ \text{by symmetry "areas" cancel}} = \int_{-5}^{5}3\,dt = \boxed{30}$$




(19) $$\int \frac{1}{\left(\sin\frac{\theta}{2} - \cos\left(\frac{\theta}{2}\right)\right)^{2}}\,d\theta = \int \frac{1}{\underbrace{\sin^{2}\frac{\theta}{2} + \cos^{2}\frac{\theta}{2}}_{1} \underbrace{- 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}_{\sin(2\cdot\frac{\theta}{2}) = \sin\theta}}\,d\theta = \int \frac{1}{1-\sin\theta}\,d\theta$$

$$= \int \frac{1}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}\,d\theta = \int \frac{1+\sin\theta}{1-\sin^{2}\theta}\,d\theta = \int \frac{1+\sin\theta}{\cos^{2}\theta}\,d\theta$$

$$= \int\left(\frac{1}{\cos^{2}\theta} + \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}\right)d\theta = \int\left(\sec^{2}\theta + \sec\theta\tan\theta\right)d\theta$$

$$= \boxed{\tan\theta + \sec\theta + C}$$

(20) $$\int \frac{(x-1)^{2}+2}{x}\,dx = \int \frac{x^{2}-2x+1+2}{x}\,dx = \int\left(x-2+\frac{3}{x}\right)dx$$

$$= \boxed{\frac{1}{2}x^{2} - 2x + 3\ln(x) + C}$$

(21) $f(x) = x^{2} - 3x + 5$ all widths are 1

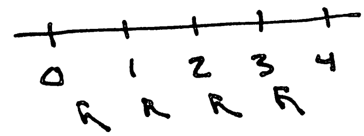
0 1 2 3 4

$$\text{Area} \approx f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1$$

$$\approx 3 + 3 + 5 + 9$$

$$\approx \boxed{20}$$

(22) Distance traveled $= \int_0^4 v(t)\,dt$




all interval widths are 1

$v(t) = \frac{5t}{t^2+1}$

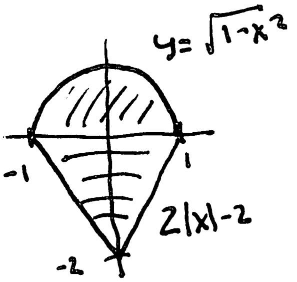
$$\int_0^4 v(t)\,dt \approx v(0)\cdot 1 + v(1)\cdot 1 + v(2)\cdot 1 + v(3)\cdot 1$$

$$\approx 0 + \frac{5}{2} + \frac{10}{5} + \frac{15}{10}$$

$$\approx \frac{5}{2} + 2 + \frac{3}{2}$$

$$\approx \boxed{6}$$

(23)

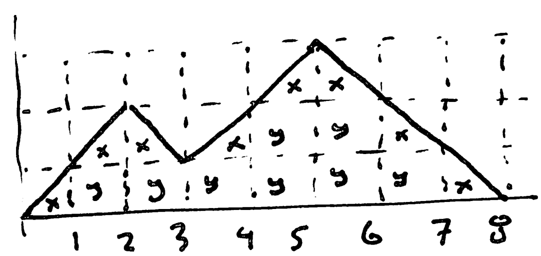



Area $= \frac{1}{2}$ of unit circle + triangle

$$= \frac{1}{2}\pi + \frac{1}{2}\cdot 2\cdot 2$$

$$= \boxed{\frac{1}{2}\pi + 2}$$

(24) $h(x) = \begin{cases} 2-|x-2| & \text{if } 0\le x\le 3 \\ 3-|x-5| & \text{if } 3\le x\le 8 \end{cases}$




$$8\cdot x + 8\cdot y = 8\cdot\frac{1}{2} + 8\cdot 1 = \boxed{12}$$

(25) @ $x=2$ $\int_2^2 tf(t)\,dt = 0 = e^0 + C\cdot 1 + 4 + 1 \leadsto C = -6$

Flip bounds & negate

$$\int_2^x tf(t)\,dt = 6\cos(\pi x) - e^{3x-6} - 2x - 1$$

Fundamental Theorem of Calculus (take derivative of both sides w.r.t. x)

$$xf(x) = -6\pi\sin(\pi x) - 3e^{3x-6} - 2$$

$$\Rightarrow \boxed{f(x) = \frac{-6\pi\sin(\pi x) - 3e^{3x-6} - 2}{x}}$$

(26) $\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h(x))\,h'(x) - f(g(x))\,g'(x)$

$\frac{d}{dx}\left(\int_{x}^{x^2} \frac{\arctan t}{t^4+1}\,dt\right) = \boxed{\frac{\arctan(x^2)}{x^8+1}\cdot 2x - \frac{\arctan(x)}{x^4+1}}$

---

(27) $\int_1^3 \frac{1}{1+(y^{1/2})^2}\cdot\frac{1}{y^{1/2}}\,dy = 2\int_1^{\sqrt{3}} \frac{1}{1+u^2}\,du = 2\arctan(u)\Big|_1^{\sqrt{3}}$

$u = y^{1/2}$

$du = \frac{1}{2}y^{-1/2}\,dy$ or $2\,du = \frac{1}{y^{1/2}}\,dy$

$= 2\arctan(\sqrt{3}) - 2\arctan(1)$

$= \frac{2\pi}{3} - \frac{2\pi}{4}$

$= \frac{8\pi}{12} - \frac{6\pi}{12} =$

$= \frac{2\pi}{12} = \boxed{\frac{\pi}{6}}$

---

(28) $\int x(\sin(x^2)+\cos(x^2))^2\sin(x^2)\,dx = \int x\big(\underbrace{\sin^2(x^2) + 2\sin(x^2)\cos(x^2) + \cos^2(x^2)}_{=1}\big)\sin(x^2)\,dx$

$= \int x\sin(x^2)\,dx + \int 2x\sin^2(x^2)\cos(x^2)\,dx$

$u = x^2$ $\qquad$ $v = \sin(x^2)$

$\frac{1}{2}du = x\,dx$ $\qquad$ $dv = \cos(x^2)\cdot 2x\,dx$

$= \int \frac{1}{2}\sin(u)\,du + \int v^2\,dv$

$= -\frac{1}{2}\cos(u) + \frac{1}{3}v^3 + C$

$= \boxed{-\frac{1}{2}\cos(x^2) + \frac{1}{3}\sin^3(x^2) + C}$

(29) first find intersection point

$$2\sin x = \sec x \tan x = \frac{\sin x}{\cos^2 x} \rightsquigarrow \cos^2 x = \frac{1}{2} \rightsquigarrow \cos x = \frac{1}{\sqrt{2}} \rightsquigarrow x = \frac{\pi}{4}$$

so area is for $0 \le x \le \frac{\pi}{4}$ note: $2\sin x$ on top of $\sec x \tan x$ in this interval

$$\text{Area} = \int_0^{\pi/4} (2\sin x - \sec x \tan x)\,dx$$

$$= (-2\cos x - \sec x)\Big|_0^{\pi/4}$$

$$= \left(-2 \cdot \frac{\sqrt{2}}{2} - \sqrt{2}\right) - (-2 - 1)$$

$$= \boxed{-2\sqrt{2} + 3}$$

---

(30) $$\text{average} = \frac{1}{2-0}\int_0^2 \frac{x}{x^2+1}\,dx = \frac{1}{2}\cdot\frac{1}{2}\int_1^5 \frac{du}{u} = \frac{1}{4}\ln u \Big|_1^5$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{4}\ln 5 - \underbrace{\frac{1}{4}\ln 1}_{=0}$$

$$= \boxed{\frac{1}{4}\ln 5}$$

---

(31) $$\frac{dy}{dt} = \frac{\cos(t)}{2y+2} \rightsquigarrow (2y+2)\,dy = \cos(t)\,dt \rightsquigarrow \int (2y+2)\,dy = \int \cos(t)\,dt$$

$$\rightsquigarrow y^2 + 2y = \sin(t) + C$$ $y(0) = -2$ so $(-2)^2 + 2(-2) = \sin(0) + C \Rightarrow C = 0$

$$\rightsquigarrow y^2 + 2y = \sin(t)$$ (now complete the square)

$$\rightsquigarrow y^2 + 2y + 1 = \sin(t) + 1$$

$$\rightsquigarrow (y+1)^2 = \sin(t) + 1$$

$$\rightsquigarrow y + 1 = -\sqrt{\sin(t)+1}$$

have to use the "-" otherwise initial condition not satisfied

$$\boxed{y = -1 - \sqrt{\sin(t)+1}}$$

(32) $\frac{dA}{dt} = \frac{26t}{A^{1/2}}$ or $A^{1/2}\,dA = 26t\,dt$

$$\int A^{1/2}\,dA = \int 26t\,dt$$

$$\frac{2}{3}A^{3/2} = 13t^2 + C \qquad A(0) = 9$$

$$18 = \frac{2}{3}\cdot 3^3 = \frac{2}{3}\cdot 9^{3/2} = 13\cdot 0^2 + C = C$$

$$\frac{2}{3}A^{3/2} = 13t^2 + 18$$

$$A^{3/2} = \frac{39}{2}t^2 + 27 \rightsquigarrow A = \left(\frac{39}{2}t^2 + 27\right)^{2/3}$$

$$A(6) = \left(\frac{39}{2}.36 + 27\right)^{2/3} = 729^{2/3} = 9^2 = \boxed{81 \text{ acres}}$$

---

(33) $\lim\limits_{t\to 0} \dfrac{\int_{3t}^{5t} \sin(x)\,dx}{t^2} \longrightarrow \dfrac{\int_0^0 \sin(x)\,dx}{0^2}$ or $\dfrac{0}{0}$

$\overset{L.H.}{=} \lim\limits_{t\to 0} \dfrac{5\sin(5t) - 3\sin(3t)}{2t} \longrightarrow \dfrac{5\sin(0) - 3\sin(0)}{2\cdot 0}$ or $\dfrac{0}{0}$

$\overset{L.H.}{=} \lim\limits_{t\to 0} \dfrac{25\cos(5t) - 9\cos(3t)}{2} = \dfrac{25-9}{2} = \dfrac{16}{2} = \boxed{8}$

---

(34) $\lim\limits_{x\to 2} \dfrac{xf(x)+4}{(x-2)^2} \longrightarrow \dfrac{2f(2)+4}{0}$ so need $2f(2)+4=0$ or $f(2) = -2$

$\overset{L.H.}{=} \lim\limits_{x\to 2} \dfrac{f(x)+xf'(x)}{2(x-2)} \longrightarrow \dfrac{f(2)+2f'(2)}{0}$ so need $-2+2f'(2)=0$ or $f'(2) = 1$

$\overset{L.H.}{=} \lim\limits_{x\to 2} \dfrac{f'(x)+f'(x)+xf''(x)}{2} = \dfrac{2f'(2)+2f''(2)}{2} = 1+f''(2) = 3$
so $f''(2) = 2$

$$\boxed{f(2) = -2 \quad f'(2) = 1 \quad f''(2) = 2}$$