Basics of derivatives

$$\frac{d}{dx}\big(x^n\big) = nx^{n-1} \qquad \frac{d}{dx}\big(\sin x\big) = \cos x$$

$$\frac{d}{dx}\big(\ln x\big) = \frac{1}{x} \qquad \frac{d}{dx}\big(\cos x\big) = -\sin x$$

$$\frac{d}{dx}\big(e^x\big) = e^x \qquad \frac{d}{dx}\big(\tan x\big) = \sec^2 x$$

$$\frac{d}{dx}\big(\arctan x\big) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\big(\sec x\big) = \sec x \tan x$$

$$\frac{d}{dx}\big(af(x) + bg(x)\big) = af'(x) + bg'(x)$$

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

$$\frac{d}{dx}\big(f(g(x))\big) = f'\big(g(x)\big)g'(x)$$

$$\frac{d}{dx}\big(f(x)\big) = f(x)\frac{d}{dx}\big(\ln\big(f(x)\big)\big)$$

$$\frac{d}{dx}\big(f^{-1}(x)\big) = \frac{1}{f'\big(f^{-1}(x)\big)}$$

Applications of derivatives

Related rates: Given some rate(s) of change, find another rate.

1. Find relationship between quantities (variables) which are changing (draw a picture; Pythagorean Theorem, similar triangles, right triangles, areas, volumes, etc.).
2. Take derivative of everything with respect to t.
3. Plug in what is known; solve for the unknown.

Linearization: Tangent lines make good local approximations to functions. At $x = a$,

$$f(x) \approx L(x) = \underbrace{f(a) + f'(a)(x-a)}_{=\text{Linearization}}$$

Absolute max/min: Max and min occur at critical points (boundary; derivative is zero; derivative is undefined). For continuous function $f(x)$ on interval $a \leqslant x \leqslant b$ find absolute max and min by following:

1. List all critical points.
2. Evaluate $f(x)$ at all points on the list.
3. Largest output is absolute max.
   Smallest output is absolute min.

Information about graph: Function is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$. Function is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. (Inflection point is where concavity changes, i.e., where second derivative changes sign.)

Classifying critical points: To use first derivative test look at first derivative above/below critical point: sign goes from positive to negative if local max; sign goes from negative to positive if local min; otherwise neither max or min. To use second derivative test look at second derivative at critical point: second derivative positive if local min; second derivative negative if local max; otherwise test is inconclusive.

Optimization: Express what is being optimized as a function of a **single** variable. Find critical points, classify the critical points, give appropriate answer (might be the input (where it happens) or output (optimized value) or both).

Approximating roots: To approximate roots (i.e., $f(x) = 0$) we take a current guess (x_n) and produce a better guess (x_{n+1}), and repeat as needed:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \text{(Newton's method)}$$

Determining indeterminates: If $\frac{f(x)}{g(x)} \to \frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x \to a$ *and* $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \qquad \text{(L'Hospital)}$$

(Don't forget to check limit is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before using.)

Algebra: Calculus is built off of algebra and you will be expected to be able to carry out basic algebra including:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\frac{a}{b} = \frac{c}{d} \quad \text{becomes} \quad ad = bc.$$

Look for cancellations and simplifications. Also there are things you should **not** do, for example $\sqrt{1-x^2} \neq 1 - x$. (If your write this on your test it might burst into flames from math blasphemy!)

Trigonometry: There are *tons* of possible trigonometry identities. But for this exam it will suffice to know the following:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Arithmetic: Arithmetic is throughout the entire test. Remember (i) you can use a calculator so if you are hesitant at all then grab your calculator and do the computation; and (ii) you do not need to simplify your answers so if you get to the answer and only have arithmetic left, stop and circle your answer. Most mistakes on exams are the result of bad arithmetic and/or bad copying.

Note: If a computation becomes crazy and/or needs a calculator, then mistakes have been made!

Basics of integrals

$$\int \frac{du}{u} = \ln u + C \qquad \int u^n\,du = \frac{1}{n+1}u^{n+1} + C$$

$$\int e^u\,du = e^u + C \qquad \int \frac{du}{1+u^2} = \arctan u + C$$

$$\int \cos u\,du = \sin u + C \qquad \int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C \qquad \int \sec u \tan u\,du = \sec u + C$$

(If the integral you are doing is *not* one of the above, then don't try to do it (you will likely be wrong)!)

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0 \qquad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b \big(cf(x) + dg(x)\big)\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$$

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Riemann sums: We can approximate the total cumulative change of $f(x)$ on the interval $a \leqslant x \leqslant b$ by breaking the interval into n (equal) parts and then approximating the change on each part:

$$\text{Total} \approx \sum_{i=1}^{n} f(x_i)\Delta_i. \qquad \text{(Riemann sum)}$$

Note that x_i is a point in the ith interval, usually left, right, or center; Δ_i is width of ith interval.

Use geometry: We can do some integrals by using geometric shapes, i.e., rectangles (e.g., $y = a$), triangles (e.g., $y = bx$ or $y = c|x|$), or circles (e.g., $y = \sqrt{1-x^2}$).

Substitution: This is the *most* important tool in working with integrals; expect to use it often. Use this to simplify integrals. Make sure to account for **every** occurrence of the old variable (i.e., the function, the "$d*$", and the bounds). If looking for what to substitute see if there is something connecting to the $d*$ and what substitution would give that as its derivative.

Areas and averages: When $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, the area between the curves is:

$$\text{Area} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

If no bounds given set $f(x) = g(x)$ and solve for intersection. Also break into parts if needed, e.g., multiple curves bounding region.
For average of $f(x)$ on $a \leqslant x \leqslant b$ we have,

$$\text{Average} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Fundamental Theorems of Calculus:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

Combining this with the chain rule and basics of integrals we have:

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x).$$

If $F(x)$ is *any* antiderivative of $f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Separable differential equations: Given an expression $y' = f(x)g(y)$ and initial condition $y(a) = b$ we can solve for $y(x)$ by carrying out the following steps:

1. Separate. Write $y' = \frac{dy}{dx}$, thought of as a ratio, and rearrange to get $\frac{1}{g(y)}\,dy = f(x)\,dx$.
2. Integrate. Take antiderivative of each side independently, only one side needs "+C".
3. Uncomplicate. Use initial condition to solve for "+C", rearrange to solve for $y(x)$.

Test notes: The final exam will have one part with eight questions. The ordering is somewhat random and does not follow the order that material was presented in class.

Calculators (including graphing calculators) are allowed for the entire final exam. Make sure to read the directions and answer everything that is asked. Show your work ("my calculator said so" is not *your* work, so does not count).

Some problems will combine several ideas. Do not be intimidated by such problems, instead break it into small manageable parts and work on each part. While the test is challenging, there are no "tricks".

If you get stuck, move to a new problem and come back later. Make sure to answer as much as you can. Keep answers *organized* and *legible*.

Other advice:

- Get plenty of sleep, and keep your daily routine as much as possible. Eat a light meal before.
- Dress comfortably, and wear layers.
- Use the restroom before the exam, if you anticipate needing to use the restroom during the exam sit near an aisle seat.
- Arrive 10-15 minutes early and settle down in your seat, do some last minute reviewing.
- Bring spare writing instruments.
- While taking the exam do not think about your grade, or how well you need to do. Focus on the exam and on the problems in front of you.
- Relax. Breathe in, breathe out. Be calm.