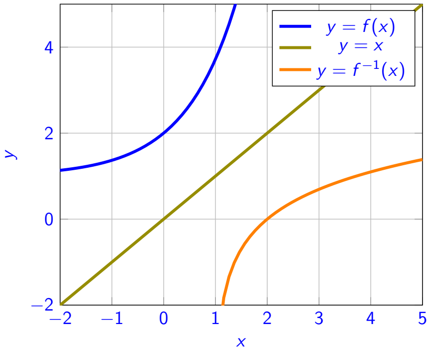
Chapter 3.8: Derivatives of Inverse Functions and Logarithms

Inverse Function



Observation: Tangent lines will also flip across the line $y = x$.

If $y = mx + b$ is a tangent line to $y = f(x)$ at $(a, f(a))$,

then $x = my + b$ is the tangent line to $y = f^{-1}(x)$ at $(f(a), a)$.

Note $x = my + b$ becomes $y = \frac{1}{m}(x - b)$.

The slope $\frac{1}{m}$ reciprocal.

Functions are related to their inverses by flipping across the line $y = x$.

Recall: f has to passes the horizontal line test to have inverse.

Derivatives of Inverse Functions

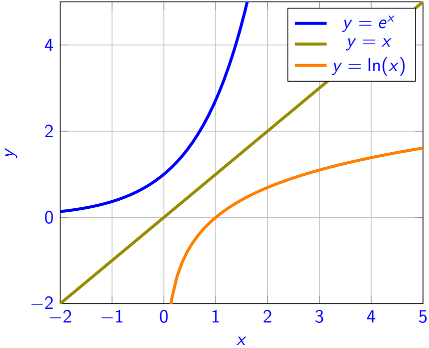
$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

Derivation can be done using implicit function approach:

$$\begin{aligned} x &= x \\ f(f^{-1}(x)) &= x \\ \frac{d}{dx}\left[f(f^{-1}(x))\right] &= \frac{d}{dx}\left[x\right] \\ f'(f^{-1}(x)) \cdot \frac{d}{dx}\left[f^{-1}(x)\right] &= 1 \\ \frac{d}{dx}\left[f^{-1}(x)\right] &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

Example: $f(x) = x^3 + 4x + 5$. Compute $f^{-1}(x)$ at $x = 10$.

Recall Exponential and Log Functions



$$f(x) = e^x \leftrightarrow f^{-1}(x) = \ln(x)$$

$$e^a e^b = e^{a+b} \leftrightarrow \ln(ab) = \ln(a) + \ln(b)$$

$$(e^a)^b = e^{ab} \leftrightarrow \ln\left(a^b\right) = b\ln(a)$$

$$\frac{e^a}{e^b} = e^{a-b} \leftrightarrow \ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$e^{\ln x} = x \leftrightarrow \ln\left(e^x\right) = x$$

$$e^1 = e \leftrightarrow \ln(e) = 1$$

$$e^0 = 1 \leftrightarrow \ln(1) = 0$$

$$a = e^{\ln a} \qquad \log_a x = \frac{\ln x}{\ln a}$$

Derivative of $\ln(x)$

Using inverse derivative

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

$$\frac{d}{dx}\left[\ln(x)\right] =$$

Using implicit function differentiation

$$y = \ln x$$
$$e^y = x$$
$$\frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(x)$$

$$\frac{d}{dx}\left[\ln(x)\right] =$$

Interesting: $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$. How could you get x^{-1} on the right-hand side?

Examples for Derivatives with ln

1. $\dfrac{d}{dx}[\ln(3x)] =$

2. $\dfrac{d}{dt}\left[t[\ln(t)]^2\right] =$

3. $\dfrac{d}{dx}\left[\dfrac{\ln(x)}{x}\right] =$

4. $\dfrac{d}{dx}[3^x] =$

5. $\dfrac{d}{dx}[x^x] =$

Logarithmic Differentiation

Idea: Instead of using

Example: $y = \sqrt{x(x+1)}$

$$y = f(x)$$
$$\frac{dy}{dx} = f'(x)$$

First take ln of both sides and then take the derivative.

$$\begin{aligned} y &= f(x) \\ \ln(y) &= \ln(f(x)) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}\left[\ln(f(x)\right] \\ \frac{dy}{dx} &= y \cdot \frac{d}{dx}\left[\ln(f(x)]\right) \\ \frac{dy}{dx} &= f(x) \cdot \frac{d}{dx}\left[\ln(f(x)]\right) \end{aligned}$$

Sometimes $\ln(f(x))$ is simpler for taking derivative than $f(x)$.

Logarithmic Differentiation Examples

Use logarithmic differentiation to compute the derivatives of following functions. Note we computed these already with different method.

$y = 3^x$ $\qquad$ $y = x^x$

Logarithmic Differentiation Examples Round II

Use logarithmic differentiation to compute the derivatives of following:

$$y = \sqrt[3]{\frac{t}{t+1}}$$

$$f(t) = t(t+1)(t+2)$$

Chapter 3.8 Recap

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

If $y = f(x)$ then

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\ln(f(x)\right]$$