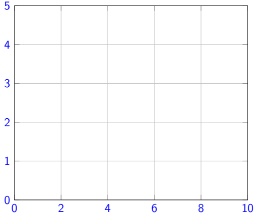
Chapter 4.2: The Mean Value Theorem

Rolle's Theorem

If $f(x)$ is a function so that

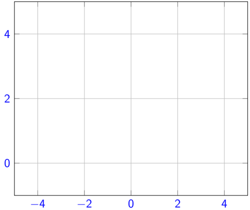
- $f(a) = f(b)$
- continuous for $a \leq x \leq b$
- differentiable for $a < x < b$

then for some c where $a < c < b$ we have $f'(c) = 0$.



Idea: f must achieve and absolute max/min. These are at critical points and at least one is not an endpoint so must be where derivative of f is 0.

Differentiable is necessary



Example: $f(x) = x^3 - 7x$, $a = -3$, $b = 1$

Mean Value Theorem

If $f(x)$ is a function that is

- continuous for $a \leq x \leq b$
- differentiable for $a < x < b$

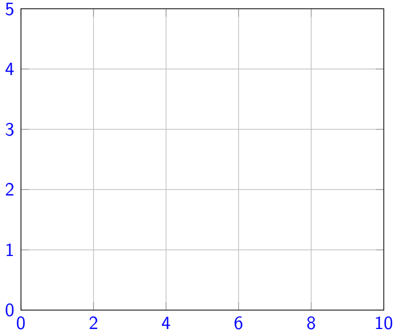
then for some c where $a < c < b$ we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Idea: Instantaneous rate of change at c is equal to the average rate of change for $a \leq x \leq b$.

Rolle's Theorem, but slightly tilted.

Example:



Note: The mean value theorem has been used by law enforcement to catch speeders!

Examples $f'(c) = \frac{f(b)-f(a)}{b-a}$ for $c \in (a, b)$

Verify the mean value theorem for $h(x) = \ln(x-1)$ for $a = 2$ and $b = e+1$

$f(x) = \frac{2x}{2x+1}$, $a = 0$, $b = 1$

Consequences of $f'(c) = \frac{f(b)-f(a)}{b-a}$ for $c \in (a, b)$

- If $f'(x) = 0$ for all $a < x < b$, then f is constant on (a, b). That is $f(x) = c$.

- If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there is a constant C such that

$$f(x) = g(x) + C$$

for all $x \in (a, b)$.

If two derivatives are equal, they came from functions which differ by a constant.

Examples

Consider

$$f(x) = \ln x \qquad g(x) = \ln(ax)$$

Show $f'(x) = g'(x)$ and determine C so that $f(x) = g(x) + C$.

Find all function $f(x)$ whose derivative is $\cos(x)$ on $(-\infty, \infty)$.

Examples

Find all function $f(x)$ whose derivative is $1/x$ on $(0, \infty)$ and $f(1) = 0$.

Given that the velocity is $v = 32t - 2$ on $(0, 1)$ and $s(1/2) = 4$, find an equation for the position function $s(t)$ for t in $(0, 1)$.