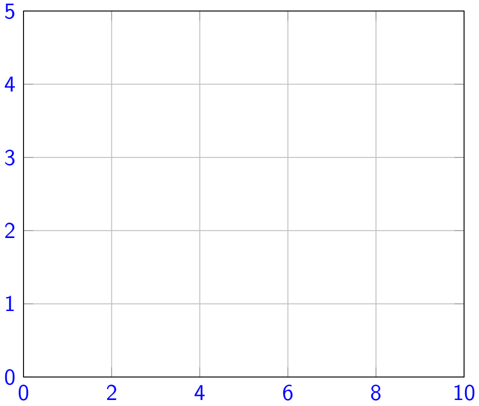
Chapter 4.7: Newton's Method

Newton's Method - Idea

Goal:
Get approximation of roots.
That is, solve $f(x) = 0$.

Idea:
It is easy to find a root of a line.
If we have a reasonable guess, we can improve it by approximating f by a tangent line.

Use: Recall, for min/max of f, we need to solve $f'(x) = 0$.



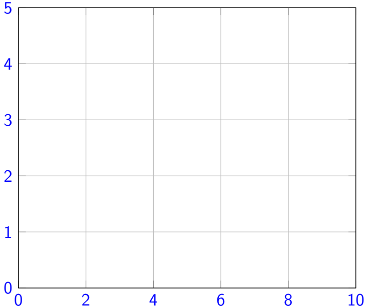
Newton's Method - Formula

Outline of the method:
Start with an initial guess and keep improving it.
Good initial guess is important!

- Start with initial guess x_0.
- Repeatedly apply the following formula to get (hopefully) better approximations.

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

- Stop when approximation is sufficient.
  Or after some number of steps, or if it starts exploding.



$$\begin{aligned} 0 &= f(a) + f'(a)(b-a) \\ -f(a) &= f'(a)(b-a) \\ \frac{-f(a)}{f'(a)} &= b - a \end{aligned}$$

$$\boxed{b = a - \frac{f(a)}{f'(a)}}$$

Example $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: Approximate $\sqrt{2}$

This method will not find the exact root, only (good) approximation.

Example $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: Approximate $2x = \cos(x)$ starting with $x_1 = 0$.

Example $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: Approximate $x^3 - x = 1$

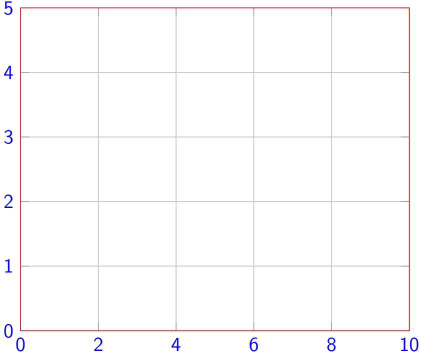
starting with $x = 1$.

starting with $x = 0$.

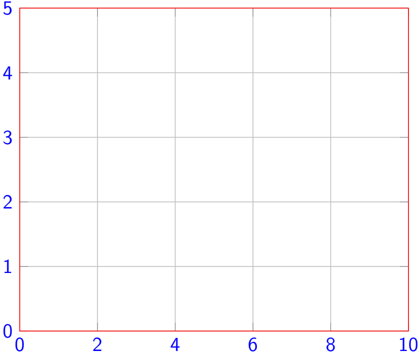
Fails

Newtons method may fail in many ways, such as division by zero, converging to a different root, not converging at all or even diverging. Initial guess is important!

Converging to a different root



Not converging at all



I promise that this is very rare and Newton's method is great!

Failing example $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Try Newton's method for

$$f(x) = \sqrt{|x|}$$

with initial guess $x_1 = 1$. Note that

$$f'(x) = \frac{1}{2\sqrt{|x|}} \cdot \frac{x}{|x|} = \frac{x}{2|x|\sqrt{|x|}}.$$