Chapter 5.3: The Definite Integral

Limits of Riemann Sums using $\int$

Computing the area under $f(x)$ for $x \in [a, b]$: Pick $a = a_0 < \cdots < a_n = b$ and $a_{k-1} \leq x_i \leq a_k$ and $\Delta_k = a_k - a_{k-1}$.

$$\text{area} = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta_k$$

Notation using the *definite integral*

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta_k$$

to b

in variable x.

Integral $\int_a^b f(x)\, dx$

from a of function $f(x)$

Few things to notice

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta_k$$

If the limit exists, f is called *integrable*.

All continuous functions and functions with finitely many jumps are integrable.

Line has an orientation $a < b$. Flipping bounds flips sign.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Recall: Area if $f(x) < 0$, the area between $f(x)$ and the axis is negative.

Easy examples

Evaluate the following integrals

- $\int_{-1}^{2} x \, dx =$

- $\int_{0}^{2} f(x) \, dx$, where $f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ x & 1 < x \leq 2 \end{cases}$

- $\int_{0}^{1} \sqrt{1 - x^2} \, dx =$

Properties of integration

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^a f(x)\,dx =$
- $\int_a^b (f(x)+g(x))\,dx =$
- $\int_a^b c \cdot f(x)\,dx =$
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx =$
- If $m \le f(x) \le M$ for $a \le x \le b$, then

More examples

Example: Find $\int_1^5 f(x)\,dx$ given

$$\int_1^3 f(x)\,dx =$$

$$\int_2^3 f(x)\,dx =$$

$$\int_2^5 f(x)\,dx =$$

Example: Given that

$$\int_4^7 f(x)\,dx =$$

$$\int_4^7 g(x)\,dx =$$

$$\int_4^7 (3 \cdot f(x) + 2 \cdot g(x))\,\,dx$$

Even more examples

$$\int_{-5}^{5} \frac{t^3}{t^4 + t^2 + 1} \, dt =$$

The *average value* of $f(x)$ on $[a, b]$ is

$$\frac{1}{b-a} \int_a^b f(x) \, dx$$

Find the average value of $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$.