Chapter 5.4: The Fundamental Theorem of Calculus

(the moment you have been all waiting for)

Mean Value Theorem Again

Let f be continuous on $[a, b]$. Then there exists a c in $[a, b]$ such that

$$f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

Idea: Use the Intermediate Value Theorem.
Let m be the minimum of $f(x)$ on $[a, b]$.
Let M be the maximum of $f(x)$ on $[a, b]$.

Cumulative $F(x)$

Let $f(x)$ be a continuous function on $[a,b]$. Define on $[a,b]$ a new *cumulative* function $F(x)$ as

$$F(x) = \int_a^x f(t)\,dt.$$

Relation of $f(x)$ and $F(X)$:

Fundamental Theorem of Calculus, Part I

Let $f(x)$ be a continuous function on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$. Then

$$F'(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Goal:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

Examples for $\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x)$

- $\dfrac{d}{dx}\left(\displaystyle\int_{10}^{x} \sin(t^4)e^t\, dt\right) =$
- $\dfrac{d}{dx}\left(\displaystyle\int_{x}^{5} 3t\sin(t)\, dt\right) =$
- $\dfrac{d}{dx}\left(\displaystyle\int_{1}^{x} (t^3+1)\, dt\right) =$
- $\dfrac{d}{dx}\left(\displaystyle\int_{x}^{5} cos(t^3) - 5\, dt\right) =$
- $\dfrac{d}{dx}\left(\displaystyle\int_{3}^{7} t^2\, dt\right) =$
- $\displaystyle\int_{3}^{x} f(t)\, dt = x^2 - 9$ Find $f(x) =$
- $\dfrac{d}{dx}\left(\displaystyle\int_{1}^{x^2} \cos(t)\, dt\right) =$

Examples for $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

- $\frac{d}{dx}\left(\int_1^{x^2} \cos(t)\,dt\right) =$

- $\frac{d}{dx}\left(\int_{x^3}^{3} \sin(t)\,dt\right) =$
- $\frac{d}{dx}\left(\int_{x^3}^{5} \sin(t)\,dt\right) =$

Chain Rule

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt\right) = f(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) =$$

Example:

$$\frac{d}{dx}\left(\int_{3x}^{e^x} \sin(t)\,dt\right)$$

Fundamental Theorem of Calculus, Part II

Let $F(x)$ be *any* antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Examples for $\int_a^b f(x)\,dx = F(b) - F(a)$

$$\int_0^2 (2x+3)\,dx =$$

- $\int_{-2}^0 2t + 5\,dt =$
- $\int_{-2}^3 2xe^{x^2}\,dx =$
- $\int_0^\pi \cos(x)\,dx =$
- $\int_0^1 \frac{1}{1+z^2}\,dz =$

Scary bonus: $\int_{-1}^1 \frac{1}{x^2}\,dx =$